

# AN $H^1$ MODEL FOR INEXTENSIBLE STRINGS

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In this paper, we are interested in local and global well-posedness for the equations of motion of inextensible strings (whips), described for  $\eta: \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}^2$  by

$$(1) \quad \eta_{tt} - \alpha^2 \eta_{ttss} = \partial_s(\sigma \eta_s),$$

with  $\sigma: \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$  determined implicitly by

$$(2) \quad -|\eta_{st}|^2 = \langle \eta_s, \partial_s(1 - \alpha^2 \partial_s^2)^{-1} \partial_s(\sigma \eta_s) \rangle$$

to ensure that  $\eta$  satisfies the inextensibility constraint  $|\eta_s| \equiv 1$ , where  $\alpha > 0$  is some given parameter. We will assume the whip is fixed at  $s = 0$  at the origin, while the end at  $s = 1$  is free; with these boundary conditions, the boundary conditions for the inverse operator  $f \mapsto (1 - \alpha^2 \partial_s^2)^{-1} f$  appearing in (2) are  $f_s(0) = 0$  and  $f(1) = 0$ . Other boundary conditions may be treated using essentially the same methods. When  $\alpha = 0$  the local well-posedness was proved by the first author [P1, P2]; the equation with  $\alpha > 0$  represents a regularization and is simpler to study despite looking superficially more complicated.

Equations (1)–(2) are critical points of the Lagrangian

$$\mathcal{L} = \frac{1}{2} \int_a^b \int_0^1 \left( |\eta_t(t, s)|^2 + \alpha^2 |\eta_{ts}(t, s)|^2 \right) ds dt$$

under the constraint  $|\eta_s(t, s)| \equiv 1$ , and they trace out geodesics in the infinite-dimensional manifold  $\mathcal{A}$  of inextensible curves with the corresponding (weak) Riemannian metric. We will show that in fact equations (1)–(2) in fact give a  $C^\infty$  ordinary differential equation on  $\mathcal{A}$ , and thus there is a  $C^\infty$  Riemannian exponential map which maps initial conditions  $(\gamma_0, v_0) \in T\mathcal{A}$  to time-one solutions  $(\eta(1), \eta_t(1))$ . As a consequence we obtain a nondegenerate Riemannian distance on  $\mathcal{A}$  and uniqueness of minimizing geodesics between sufficiently close curves, and we can use curvature computations rigorously to understand stability of solutions.

Solutions of (1)–(2) conserve the  $H^1$  energy:

$$(3) \quad \int_0^1 |\eta_t(t, s)|^2 + \alpha^2 |\eta_{st}(t, s)|^2 ds = \text{const},$$

which we use to prove global existence for all solutions. This is in sharp contrast to the equations with  $\alpha = 0$ , where blowup seems to be fairly typical though not yet proven rigorously (see Thess et al. [TZN] for heuristics and numerical results). For an  $L^2$  whip we expect the velocity to approach infinity somewhere (which leads to the audible crack), while (3) and the Sobolev inequality imply an absolute maximum on the pointwise velocity of an  $H^1$  whip. In Figure 1 we display a blowup scenario first proposed by [TZN]: as small loops close off and the whip is suddenly forced to change direction, kinks are formed and the curvature blows up. In Figure 2 we see how the  $H^1$  metric stabilizes the evolution.

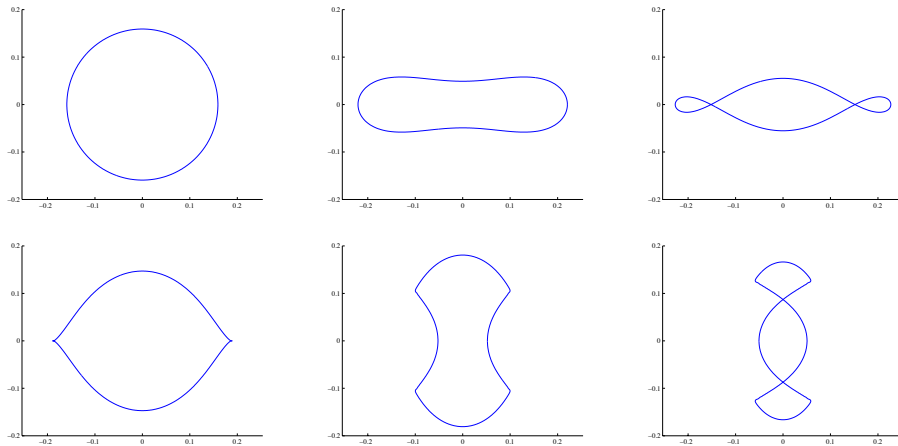


FIGURE 1. Evolution of the string in the  $L^2$  metric, for initial conditions  $\eta(0, s) = \frac{1}{2\pi}e^{2\pi is}$  and  $\eta_t(0) = 0.025(3e^{-2\pi is} + e^{6\pi is})$ , plotted at equal time steps  $t = 0.95k$  for  $0 \leq k \leq 5$ , as in [TZN]. At step  $k = 3$  the string develops cusps and hence the PDE seems to blow up. For numerical simulations, we use the chain model from [P1] with  $n = 1000$  links (using the standard “ode45” routine in MATLAB), which allows us to continue the solution past the singularity, where weak shocks appear to form.

We have several motivations for studying the system (1)–(2). First of all, it is a regularization of the  $L^2$  whip equations (with  $\alpha = 0$ ) in much the same way that the Camassa-Holm equation is a regularization of the Burgers’ equation [M, CK] and the Lagrangian-averaged Euler- $\alpha$  equations are a regularization of the Euler equations for an

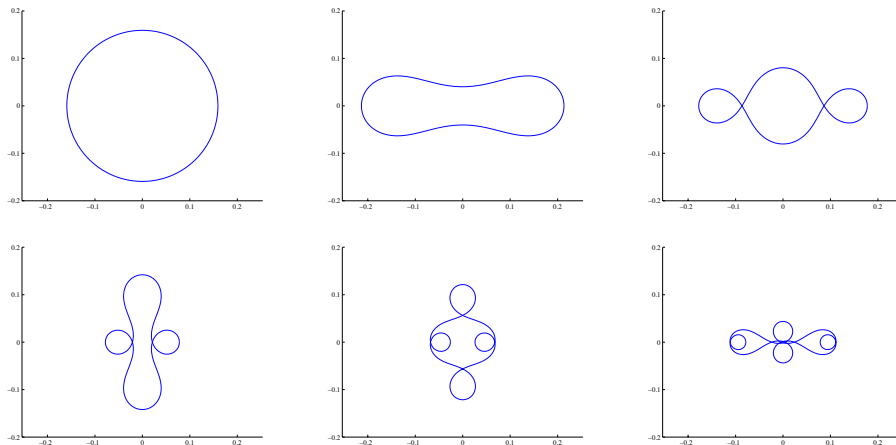


FIGURE 2. The same plot as in Figure 1, using the  $H^1$  metric with  $\alpha = 1$ . Small loops do not get pinched off and weak shocks do not develop.

ideal fluid [Shk, HL]; as in these examples, the equations are analytically simpler and have solutions with smoother dependence on initial conditions which are less likely to blow up. The associated geometry is also simpler than the geometry of the  $L^2$  equations [P2], in many of the same ways that the geometry of the space of unparametrized curves is simpler in  $H^1$  than in  $L^2$  (see Michor and Mumford [MM]); thus it gives another candidate for a distance between curves in the plane, which has application to shape recognition problems. Next, the constraint  $|\eta_s| \equiv 1$  is analogous to the volume-preserving constraint in ideal fluid mechanics, with the tension  $\sigma$  determined here in the same implicit way as the pressure is determined there; thus it gives another “toy” model for ideal fluid mechanics, which is also described by a  $C^\infty$  Riemannian exponential map (see Ebin and Marsden [EM]). Finally, if  $\sigma$  in (1) is given by  $\sigma = S(|\eta_s|)$  for some function  $S$  rather than by (2), we obtain the Pochhammer-Chree equation [S], and hence we can view the system as a geometric limiting case of a very strong force such as  $\sigma = k(|\eta_s|^2 - 1)^2$  for  $k \gg 1$ , in the same way as the Euler equations for incompressible fluids are a limiting case of the equations of compressible fluid mechanics with a strong constraining force (see Ebin [E]).

## 1. GEOMETRY

In this section we will derive the equations (1)–(2) as the geodesic equation on the Banach manifold  $\mathcal{A}$  of unit-speed curves with one end

fixed at the origin. We will use concepts from infinite-dimensional Riemannian geometry freely, based primarily on Lang [L] and Ebin-Marsden [EM]. A reader unfamiliar with Riemannian geometry may skip this section as it is primarily for motivation.

We begin by discussing the geometry of the flat ambient space. Let  $I = [0, 1]$ , and let

$$(4) \quad \mathcal{X} = \{\gamma \in C^1(I, \mathbb{R}^2) \mid \gamma(0) = 0\}.$$

This is a Banach space with norm given by  $\|\gamma\| = \sup_s |\gamma(s)| + \sup_s |\gamma'(s)|$ , or equivalently by  $\|\gamma\| = \sup_s |\gamma'(s)|$  since  $\gamma(0) = 0$  obviously implies  $\sup_s |\gamma(s)| \leq \sup_s |\gamma'(s)|$ . Since  $\mathcal{X}$  is linear, we can view it as a smooth manifold with the tangent spaces all isomorphic to  $\mathcal{X}$ , i.e., we have  $T_\gamma \mathcal{X} = \{v \in C^1(I, \mathbb{R}^2) \mid v(0) = 0\}$ .

We define a weak Riemannian metric on  $\mathcal{X}$  by the formula

$$(5) \quad \langle\langle v, v \rangle\rangle_\gamma = \int_0^1 \langle v(s), v(s) \rangle + \alpha^2 \langle v'(s), v'(s) \rangle ds.$$

(The metric is called weak because the  $H^1$  topology it generates is weaker than the  $C^1$  topology we will use.) Weak metrics do not always have Levi-Civita connections [MEF], but if one exists then it must be unique. In this case it is easy to see that there is a connection, and it is completely determined by how it covariantly differentiates a vector field along a curve. Specifically, let  $\gamma$  be a curve in  $\mathcal{X}$  and  $w$  be a vector field along  $\gamma$  (i.e.,  $\gamma(t, s)$  and  $w(t, s)$  are  $C^\infty$  in  $t$  and  $C^1$  in  $s$ , with  $\gamma(t, 0) = w(t, 0) = 0$  for all  $t$ ); then the covariant derivative of  $w$  along  $\gamma$  is

$$(6) \quad \frac{Dw}{dt}(t, s) = \frac{\partial w}{\partial t}(t, s).$$

As a consequence the geodesic equation is  $\frac{D}{dt} \frac{d\eta}{dt} = 0$  which reduces to  $\eta_{tt} = 0$ , so that every geodesic is  $\eta(t, s) = \gamma(s) + tv(s)$  for some  $C^1$  initial position  $\gamma$  and velocity  $v$ . In other words, geodesics in  $\mathcal{X}$  are simply families of straight lines in  $\mathbb{R}^2$ . It is easy to compute (e.g., using the Jacobi equation) that the Riemann curvature tensor vanishes, and thus the space  $\mathcal{X}$  is flat.

Now the space we are actually interested in is the space  $\mathcal{A}$  of unit-speed arcs, which we will define in terms of the angle function: if  $\gamma$  is a  $C^1$  arc with  $|\gamma'(s)| = 1$  for all  $s$  and  $\gamma(0) = 0$ , then

$$(7) \quad \gamma(s) = \left( \int_0^s \cos \theta(x) dx, \int_0^s \sin \theta(x) dx \right)$$

for some uniquely determined continuous function  $\theta: I \rightarrow S^1$ . (We could put  $\theta: I \rightarrow \mathbb{R}$  but then  $\theta$  is determined only up to multiples

of  $2\pi$ .) The space  $\mathcal{A} = C(I, S^1)$  of all continuous angle functions is a Banach manifold modeled on  $C(I, \mathbb{R})$  with the standard supremum norm; coordinate charts are essentially just choices of representation of the angle in the reals.<sup>1</sup>

**Theorem 1.1.** *The map  $F: \mathcal{A} \rightarrow \mathcal{X}$  given by (7) defines a smooth embedding of  $\mathcal{A}$  as a submanifold of  $\mathcal{X}$ .*

*Proof.* We first show that  $F$  is  $C^\infty$ , i.e., it has infinitely many Fréchet derivatives. The first derivative in the direction of a tangent vector  $\omega \in T_\gamma \mathcal{A}$  is

$$(8) \quad TF_\theta(\omega)(s) = \left( - \int_0^s \omega(x) \sin \theta(x) dx, \int_0^s \omega(x) \cos \theta(x) dx \right),$$

which for any fixed  $\theta$  is a continuous linear map from the space  $T_\theta \mathcal{A} = C(I, \mathbb{R})$  to  $T_{F(\theta)} \mathcal{X} = C^1(I, \mathbb{R}^2)$  (with norm 1). Furthermore as a function on  $T\mathcal{A} \simeq C(I, S^1) \times C(I, \mathbb{R})$ , the map  $(\theta, \omega) \mapsto TF_\theta(\omega)$  is clearly continuous in the supremum topology, which establishes that  $F$  is  $C^1$ . Showing that  $F$  is  $C^k$  for every  $k > 1$  is similar.

We now show that  $F$  embeds  $\mathcal{A}$  as a Banach submanifold of  $\mathcal{X}$ ; an easy way to see this is to let  $\mathcal{X}_0$  denote the open subset of  $\mathcal{X}$  consisting of curves whose tangent vectors  $\gamma'(s)$  are nowhere zero; any such curve  $\gamma$  can be written uniquely as

$$\gamma(s) = \left( \int_0^s e^{\phi(x)} \cos \theta(x) dx, \int_0^s e^{\phi(x)} \sin \theta(x) dx \right)$$

for continuous functions  $\phi: I \rightarrow \mathbb{R}$  and  $\theta: I \rightarrow S^1$ , and the subset  $\mathcal{A}$  is just the set of all such  $\gamma$  for which  $\phi \equiv 0$ . Hence we have local coordinate charts in which the image of  $F$  is a closed subspace which splits, which is precisely what we need to have a Banach submanifold; see Lang [L].  $\square$

Since  $\mathcal{A}$  is a smooth submanifold of the flat space  $\mathcal{X}$ , we have an induced Riemannian metric on  $\mathcal{A}$  given by the usual formula  $\langle \langle \omega, \omega \rangle \rangle_\theta = \langle \langle TF_\theta(\omega), TF_\theta(\omega) \rangle \rangle_\theta$ , where  $TF$  is given by (8). It is easy to compute, by rearranging the order of integration, that this metric is given

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<sup>1</sup>Specifically, for any fixed  $\theta_0 \in S^1$ , the set of all maps  $\theta \in \mathcal{A}$  with  $\theta(0) \neq \theta_0$  is open in the  $C^0$  topology, and by choosing a specific real value for  $\theta(0)$  in an open interval of length  $2\pi$  we obtain a representation of the entire function  $\theta(s)$  into the reals.

explicitly by

$$(9) \quad \langle \langle \omega, \omega \rangle \rangle_\theta = \alpha^2 \int_0^1 \omega(s)^2 ds + \int_0^1 \int_0^1 (1 - \max\{s, x\}) \cos(\theta(x) - \theta(s)) \omega(s) \omega(x) dx ds.$$

We could derive all the geometry from this Riemannian metric directly, but it is easier to use submanifold geometry since we have a flat ambient space. This is particularly true when studying the Levi-Civita connection  $\mathcal{A}$ , which is the tangential projection of the Levi-Civita connection on  $\mathcal{X}$ ; hence our next objective is to compute this projection.

For this purpose, let us define linear operators  $M_\theta$  and  $N_\theta$  from  $C(I, \mathbb{R})$  to itself, for any  $\theta \in C(I, S^1)$ :

$$(10) \quad (M_\theta h)(s) = \int_0^1 G(s, x) h(x) \cos(\theta(x) - \theta(s)) dx,$$

$$(11) \quad (N_\theta h)(s) = \int_0^1 G(s, x) h(x) \sin(\theta(x) - \theta(s)) dx,$$

where  $G$  is the Green function for  $(1 - \alpha^2 \partial_s^2)^{-1}$  with boundary conditions  $G_s(0, x) = G(1, x) = 0$ , given explicitly by

$$(12) \quad G(s, x) = \frac{1}{\alpha \cosh(1/\alpha)} \begin{cases} \cosh \frac{s}{\alpha} \sinh \frac{1-x}{\alpha} & s \leq x, \\ \sinh \frac{1-s}{\alpha} \cosh \frac{x}{\alpha} & s \geq x. \end{cases}$$

Clearly the operator norm of  $M_\theta$  is

$$\|M_\theta\|_{C(I, \mathbb{R})} = \sup_{s \in I} \int_0^1 G(s, x) dx = 1 - \frac{1}{\cosh \frac{1}{\alpha}}$$

which is strictly less than 1; as a result we know  $1 - M_\theta$  is invertible from  $C(I, \mathbb{R})$  to itself, and that

$$(13) \quad \|(1 - M_\theta)^{-1}\|_{C(I, \mathbb{R})} \leq \frac{1}{1 - \|M_\theta\|_{C(I, \mathbb{R})}} = \cosh \frac{1}{\alpha}.$$

**Theorem 1.2.** *For each  $\theta \in \mathcal{A}$ , the tangent space to  $\mathcal{X}$  at  $\gamma = F(\theta)$  splits into two subspaces which are closed in the  $C^1$  topology and orthogonal in the metric (5), given by*

$$(14) \quad T_{F(\theta)} \mathcal{X} = TF_\theta[T_\theta \mathcal{A}] \oplus \left\{ D(1 - \alpha^2 D^2)^{-1}(h\gamma'), h \in C(I, \mathbb{R}) \right\}.$$

The orthogonal projection  $\pi: T\mathcal{X}|_{F[\mathcal{A}]} \rightarrow TF[T\mathcal{A}]$  is given for any  $\gamma = F(\theta)$  and any  $w \in T_\gamma \mathcal{X}$  by

$$(15) \quad \pi_\gamma(w) = TF_\theta(j), \quad \text{where } j = \langle w', R\gamma' \rangle + N_\theta(1 - M_\theta)^{-1} \langle w', \gamma' \rangle,$$

with  $R$  denoting the two-dimensional rotation operator  $R\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$ .

The tangential projection  $\pi_\gamma(w)$  is smooth as a map from the sub-bundle  $T\mathcal{X}|_{F[\mathcal{A}]}$  to the image  $TF[T\mathcal{A}] \subset T\mathcal{X}$ .

*Proof.* First we show that every  $u = \partial_s(1 - \alpha^2 \partial_s^2)^{-1}(h\gamma')$  for  $h \in C(I, \mathbb{R})$  is orthogonal to every  $v = TF_\theta(j)$  for  $j \in C(I, \mathbb{R})$ . Fix such an  $h$  and  $j$ , and let  $\xi = (1 - \alpha^2 \partial_s^2)^{-1}(h\gamma')$ , so that  $\xi$  solves

$$\xi(s) - \alpha^2 \xi''(s) = h(s)\gamma'(s), \quad \xi'(0) = 0, \quad \xi(1) = 0;$$

then we easily see that  $\xi = (M_\theta h)\gamma' + (N_\theta h)R\gamma'$ , where  $M$  and  $N$  are defined as in (10)–(11), and we have  $u = \xi'$  with  $u(0) = 0$ . Therefore

$$\begin{aligned} \langle\langle u, v \rangle\rangle &= \int_0^1 \langle \xi'(s), v(s) \rangle ds + \alpha^2 \int_0^1 \langle \xi''(s), v'(s) \rangle ds \\ &= \langle \xi(1), v(1) \rangle - \langle \xi(0), v(0) \rangle - \int_0^1 \langle \xi(s) - \alpha^2 \xi''(s), v'(s) \rangle ds \\ &= - \int_0^1 h(s)j(s) \langle \gamma'(s), R\gamma'(s) \rangle ds = 0. \end{aligned}$$

Hence the two spaces in the direct sum in (14) are orthogonal in the  $H^1$  metric (5). To prove their union is the entire tangent space  $T_\gamma\mathcal{X}$ , we compute the orthogonal projection. Assuming that  $w = v + \xi'$  where  $v = TF_\theta(j)$  and  $\xi = (1 - \alpha^2 \partial_s^2)^{-1}(h\gamma')$ , for some continuous functions  $h$  and  $j$ , we must have  $\langle v', \gamma' \rangle = 0$ , so that  $\langle w', \gamma' \rangle = \langle \xi'', \gamma' \rangle$ . We have  $\alpha^2 \xi'' = \xi - h\gamma'$ , so that  $h$  must satisfy

$$\alpha^2 \langle w', \gamma' \rangle = -h + \langle \xi, \gamma' \rangle = -h + M_\theta h.$$

Certainly this equation can always be solved for  $h$  since  $(1 - M_\theta)$  is invertible by (13). Conversely, if we define  $h = -\alpha^2(1 - M_\theta)^{-1}(\langle w', \gamma' \rangle)$ , construct  $\xi = (1 - \alpha^2 \partial_s^2)^{-1}(h\gamma')$ , and set  $v = w - \xi'$ , we can easily see that  $\langle v', \gamma' \rangle = 0$ , and in fact  $v = TF_\theta(j)$  where  $j$  is given as in (15). Clearly both  $h$  and  $j$  will be continuous as long as  $\theta$  and  $w'$  are continuous, so we have a well-defined orthogonal projection for each fixed  $\theta$ .

Finally to establish smoothness, we just observe that, for any fixed  $f \in C(I, \mathbb{R})$ , the functions  $M_\theta f$  and  $N_\theta f$  are clearly  $C^\infty$  as functions of  $\theta \in C(I, \mathbb{R})$ , since they depend on  $\theta$  only through composition with smooth functions. Hence  $(1 - M_\theta)^{-1}$  is also smooth as a function of  $\theta$ , and the composition  $N_\theta(1 - M_\theta)^{-1}$  is smooth as a function of  $\theta$ . Smoothness in  $w$  is obvious by linearity.  $\square$

As a consequence we obtain the Levi-Civita connection in  $\mathcal{A}$ , for which smoothness is an automatic consequence of smoothness of the orthogonal projection.

**Corollary 1.3.** *The manifold  $\mathcal{A}$  with the weak Riemannian metric (9) has a smooth Levi-Civita connection, which can be described in terms of the covariant derivative as follows: let  $\theta$  be a curve in  $\mathcal{A}$  with  $\omega$  a vector field along  $\theta$ ; then the covariant derivative of  $\omega$  in the direction of  $\theta_t$  is*

$$(16) \quad \frac{D\omega}{dt} = \frac{\partial\omega}{\partial t} - N_\theta(1 - M_\theta)^{-1}(\omega\theta_t).$$

As a consequence the geodesic equation on  $\mathcal{A}$  is

$$(17) \quad \theta_{tt} = N_\theta(1 - M_\theta)^{-1}(\theta_t^2),$$

and this is a smooth second-order ordinary differential equation on  $\mathcal{A}$ .

*Proof.* The only thing to do is to observe that the corresponding vector field along  $\gamma = F(\theta)$  is  $z = TF_\theta(\omega)$ , so that  $z_s = \omega R\gamma_s$  where  $R$  is the planar rotation operator. The covariant derivative of  $z$  in the flat space  $\mathcal{X}$  is just  $z_t$ , and we have  $z_{ts} = \omega_t R\gamma_s + \omega R\gamma_{ts}$ . Since  $\gamma_s = (\cos\theta, \sin\theta)$ , we have  $\gamma_{ts} = \theta_t(-\sin\theta, \cos\theta) = \theta_t R\gamma_s$ , and thus

$$(18) \quad z_{ts} = \omega_t R\gamma_s - \omega\theta_t\gamma_s,$$

using the fact that  $R^2 = -1$ . Now the covariant derivative in  $\mathcal{A}$  is the tangential projection of the covariant derivative in  $\mathcal{X}$ , which we computed in Theorem 1.2. We apply (15) with  $w = z_t$  and use (18) to get

$$\pi_{F(\theta)}(z_t) = TF_\theta(\omega_t - N_\theta(1 - M_\theta)^{-1}(\omega\theta_t)),$$

and under the identification of  $TA$  with  $TF[TA]$  we obtain the covariant derivative in  $TA$  directly as (16).

The geodesic equation (17) is an easy consequence of the general geodesic equation  $\frac{D}{dt}\frac{d\theta}{dt} = 0$ , plugging in  $\omega = \theta_t$ .  $\square$

## 2. LOCAL AND GLOBAL EXISTENCE

In this section we analyze equation (17) as a second-order ordinary differential equation for  $\theta \in C(I, S^1)$ . On the tangent bundle  $TC(I, S^1) \cong C(I, S^1) \times C(I, \mathbb{R})$ , we can write it as the first-order system

$$(19) \quad \frac{d}{dt} \begin{pmatrix} \theta \\ \omega \end{pmatrix} = \begin{pmatrix} \omega \\ N_\theta(1 - M_\theta)^{-1}\omega^2 \end{pmatrix},$$

where  $M_\theta$  and  $N_\theta$  are given as in (10)–(11). We have already seen that the right side of this equation is  $C^\infty$  as a function of  $(\theta, \omega)$ , which means that local existence follows from the Picard iteration argument. Global existence is only slightly more involved; the main thing is to establish conservation of energy. We will work here with the equations directly,

rather than relying on the geometrical results from the previous section, in order to make this Section more accessible to a differential equations audience.

First we show that the system (19) is equivalent to the system (1)–(2). Differentiating equation (1) and setting  $L = I - \alpha^2 \partial_s^2$  gives

$$(20) \quad L\eta_{stt} = (\sigma\eta_s)_{ss} = \alpha^{-2}(I - (I - \alpha^2 \partial_s^2))(\sigma\eta_s) = \alpha^{-2}(I - L)(\sigma\eta_s),$$

and therefore

$$(21) \quad \eta_{stt} = \alpha^{-2}(L^{-1} - I)(\sigma\eta_s).$$

Identifying  $\eta_s(t, s)$  with  $\exp(i\theta(t, s))$  then gives

$$(22) \quad \alpha^2(i\theta_{tt} - \theta_t^2) = \exp(-i\theta)L^{-1}(\sigma \exp i\theta) - \sigma,$$

and so

$$(23) \quad \alpha^2(i\omega_t - \omega^2) = M_\theta\sigma + iN_\theta\sigma - \sigma$$

which is equivalent to (19).

Now we prove local existence. As mentioned, this follows from smoothness, but here we provide an explicit Lipschitz constant.

**Theorem 2.1.** *For any initial conditions  $\theta_0 \in C(I, S^1)$  and  $\omega_0 \in C(I, \mathbb{R})$ , there is an  $\varepsilon > 0$  such that there is a unique solution of the system (19) in  $C^1((-\varepsilon, \varepsilon), C(I, S^1) \times C(I, \mathbb{R}))$ .*

*Proof.* To use a Picard iteration, we prove the right side of (19) is Lipschitz. This follows from the fact that  $M_\theta$  and  $N_\theta$  are locally Lipschitz: for any fixed function  $h \in C(I, \mathbb{R})$  and  $\theta_1, \theta_2 \in C(I, S^1)$ , we have

$$\begin{aligned} |(M_{\theta_1}h - M_{\theta_2}h)(s)| &\leq \int_0^1 G(s, x) \left| \cos(\theta_1(x) - \theta_1(s)) \right. \\ &\quad \left. - \cos(\theta_2(x) - \theta_2(s)) \right| |h(x)| dx \\ &\leq \int_0^1 G(s, x) |\theta_1(x) - \theta_2(x)| |h(x)| dx \\ &\quad + |\theta_1(s) - \theta_2(s)| \int_0^1 G(s, x) |h(x)| dx, \end{aligned}$$

where  $G$  is given by (12). Taking the supremum over  $s$  and using  $\sup_s \int_0^1 G(s, x) dx = 1 - 1/\cosh(1/\alpha)$ , we obtain

$$\|M_{\theta_1}h - M_{\theta_2}h\| \leq 2 \left( 1 - \frac{1}{\cosh(1/\alpha)} \right) \|h_1\| \|\theta_1 - \theta_2\|.$$

We easily get the exact same estimate for  $N_\theta$  as well.

The formula  $(1 - M_{\theta_1})^{-1} - (1 - M_{\theta_2})^{-1} = (1 - M_{\theta_1})^{-1}(M_{\theta_1} - M_{\theta_2})(1 - M_{\theta_2})^{-1}$  implies a Lipschitz estimate for  $(1 - M_{\theta})^{-1}$ , using the fact that  $\|(1 - M_{\theta})^{-1}\| \leq \cosh(1/\alpha)$  for any  $\theta$ . Using the triangle inequality we then obtain

$$\begin{aligned} \|N_{\theta_1}(1 - M_{\theta_1})^{-1}\omega_1^2 - N_{\theta_2}(1 - M_{\theta_2})^{-1}\omega_2^2\| &\leq 2(C_\alpha + C_\alpha^2)\|\omega_1\|^2\|\theta_1 - \theta_2\| \\ &\quad + C_\alpha(\|\omega_1\| + \|\omega_2\|)\|\omega_1 - \omega_2\|, \end{aligned}$$

where  $C_\alpha = \cosh(1/\alpha) - 1$  and all norms are the supremum norms of continuous functions. Hence the right side of (19) is Lipschitz, and the fundamental theorem of ordinary differential equations in Banach spaces implies local existence on some time interval  $(-\varepsilon, \varepsilon)$ .  $\square$

We automatically get continuous dependence on the initial conditions using the standard technique, and smooth dependence follows from computing Fréchet derivatives as in the last Section.

Now we prove global existence. The essential tool is an  $L^2$  bound on  $\omega$  which comes from conservation of kinetic energy, a fact which is true for any geodesic equation. Again we will give a direct proof. To do so we define linear operators  $J_\theta$  and  $K_\theta$  from  $C(I, \mathbb{R})$  to itself as follows: let  $H(s, x) = 1 - \max\{s, x\}$ , the Green function satisfying  $H_{ss}(s, x) = -\delta(s - x)$  with boundary conditions  $H_s(0, x) = 0$  and  $H(1, x) = 0$ . Then for any continuous  $h$  we set

$$(24) \quad (J_\theta h)(s) = \int_0^1 H(s, x) \sin(\theta(x) - \theta(s)) h(x) dx,$$

$$(25) \quad (K_\theta h)(s) = \int_0^1 H(s, x) \cos(\theta(x) - \theta(s)) h(x) dx.$$

Note the similarity to (10)–(11); the Green functions have the same boundary conditions, and only the differential operator has changed from  $(1 - \alpha^2 \partial_s^2)$  to  $(-\partial_s^2)$ . Observe that the Riemannian metric (9) is

$$\langle\langle \omega, \omega \rangle\rangle_\theta = \int_0^1 \omega(K_\theta + \alpha^2)\omega ds,$$

so we expect this to be constant in time for a solution of (19).

**Theorem 2.2.** *Let  $(\theta, \omega)$  be a solution of (19) on some time interval, and define*

$$(26) \quad E(t) = \int_0^1 \omega(t, s)(K_\theta \omega)(t, s) + \alpha^2 \omega(t, s)^2 ds.$$

*Then  $E(t)$  is constant in time.*

*Proof.* The first step is to establish the formula

$$(27) \quad (K_\theta + \alpha^2)N_\theta = J_\theta(1 - M_\theta).$$

This comes from the following computation: for any angle function  $\theta \in \mathcal{A}$ , let  $\gamma = F(\theta)$  be the image as a unit-speed curve, so that  $\gamma' = e^{i\theta}$ . Take an arbitrary continuous function  $e$  and set  $\xi = (1 - \partial_s^2)^{-1}(e\gamma')$  with boundary conditions  $\xi'(0) = 0$  and  $\xi(1) = 0$ . By definition of  $M$  and  $N$ , we have  $\xi = (M_\theta e + iN_\theta e)\gamma'$ .

Define  $f$  and  $g$  by  $-\xi'' = (f + ig)\gamma'$ ; then we have

$$\xi = (K_\theta + iJ_\theta)(f + ig)\gamma'$$

by definition of  $J$  and  $K$ , and matching components we obtain the equations

$$(28) \quad M_\theta e = K_\theta f - J_\theta g \quad \text{and} \quad N_\theta e = K_\theta g + J_\theta f.$$

But we also have

$$\xi - \alpha^2 \xi'' = e\gamma' + \xi + \alpha^2 f\gamma' + \alpha^2 g\gamma',$$

from which we conclude that

$$(29) \quad e = M_\theta e + \alpha^2 f \quad \text{and} \quad 0 = N_\theta e + \alpha^2 g.$$

Combining (28) and (29) we obtain (27).

Differentiating (26) and using symmetry of  $K_\theta$ , we obtain

$$\begin{aligned} E'(t) &= \int_0^1 2\omega(K_\theta + \alpha^2)\omega_t + \omega \frac{\partial K_\theta}{\partial t}(\omega) ds \\ &= 2 \int_0^1 \omega J_\theta(\omega^2) ds + \omega \frac{\partial K_\theta}{\partial t} \omega ds, \end{aligned}$$

using (19) and (27). It is easy to compute that for any continuous  $f$  we have  $\frac{\partial K_\theta}{\partial t} f = -J_\theta(\theta_t f) + \theta_t J_\theta f$ , and plugging this in gives

$$E'(t) = \int_0^1 \omega J_\theta(\omega^2) + \omega^2 J_\theta(\omega) ds = 0$$

since  $J_\theta$  is antisymmetric. □

The operator  $K_\theta$  is clearly positive-definite in  $L^2(I, \mathbb{R})$  since we can write

$$\begin{aligned} \langle \langle f, K_\theta f \rangle \rangle_{L^2} &= \int_0^1 f(s)(K_\theta f)(s) ds \\ &= \int_0^1 \int_0^1 f(s)f(x)(1 - \max\{s, x\}) \langle \gamma'(s), \gamma'(x) \rangle dx ds \\ &= \int_0^1 \left\langle \int_0^y f(x)\gamma'(x) dx, \int_0^y f(s)\gamma'(s) ds \right\rangle dy. \end{aligned}$$

(This formula is our motivation for defining  $K_\theta$ .) Thus conservation of energy (26) implies that

$$(30) \quad \int_0^1 \omega(t, s)^2 ds \leq \frac{E_0}{\alpha^2}$$

for any solution of (19).

**Theorem 2.3.** *For any initial conditions  $\theta_0 \in C(I, S^1)$  and  $\omega_0 \in C(I, \mathbb{R})$ , the solution of the system (19) is defined for all time.*

*Proof.* This relies on the general characterization of blowup for ODEs in Banach space: the solution can be extended to a maximal interval  $(a, b)$ , and if  $b < \infty$ , then  $\lim_{t \rightarrow b^-} \|F(\theta(t), \omega(t))\| = \infty$ ; see [H]. For our equation, we have

$$(31) \quad \frac{d}{dt} \|\omega\|_{L^\infty} \leq \|N_\theta(1 - M_\theta)^{-1}\omega^2\|_{L^\infty}.$$

By (30), the  $L^1$  norm of  $\omega^2$  is always bounded by  $E_0/\alpha^2$  where  $E_0$  is the initial energy. We will now view  $M_\theta$  as an operator from  $L^1(I)$  to  $L^1(I)$ , and  $N_\theta$  as an operator from  $L^1(I)$  to  $L^\infty$ . It is easy to see that the operator norm of  $M_\theta$  in  $L^1$  is exactly the same as it is in  $L^\infty$ , and thus bounded by  $1 - 1/\cosh(1/\alpha)$  as in (13), so that  $(1 - M_\theta)^{-1}$  has operator norm at most  $\cosh(1/\alpha)$  on  $L^1$ .

For  $N_\theta$ , we have for any  $f \in L^1$  and any continuous  $\theta$  that

$$\|N_\theta f\|_{L^\infty} \leq \sup_{s \in I} \int_0^1 G(s, x) |f(x)| dx \leq \sup_{s \in I, x \in I} G(s, x) \|f\|_{L^1},$$

and the supremum of the Green function can be easily computed from the explicit formula (12) to be

$$\|N_\theta\|_{(L^1, L^\infty)} \leq \sup_{s \in I, x \in I} G(s, x) = \frac{1}{\alpha} \tanh \frac{1}{\alpha}.$$

Putting these norm estimates together in (31), we get

$$\|\omega(t)\|_{L^\infty} \leq \|\omega_0\| + \frac{E_0 t}{\alpha^3} \sinh \frac{1}{\alpha}.$$

Integrating this gives a similar estimate for  $\|\theta(t)\|_{L^\infty}$ . Hence  $(\theta, \omega)$  cannot blow up and thus neither can  $F(\theta, \omega)$ .  $\square$

The mechanism for blowup in the  $L^2$  case (when  $\alpha = 0$ ) seems to be small loops pinching into cusps (see Thess et al. [TZN] and Figure 1). When  $\alpha > 0$  such loops cannot close off, since at a cusp  $\theta$  would become discontinuous. Instead loops that form essentially have nowhere to go, and thus certain initial conditions can produce lots of small loops; see Figure 2. One could analyze this phenomenon by looking at the corresponding ODE for  $\theta_s$  and investigating the growth of the total curvature  $\int_0^1 |\theta_s| ds$  in time. However we will leave this issue aside.

### 3. OTHER ASPECTS

In this final section we discuss some extensions and applications of the results in the last two sections. In particular we show that although much of the analysis extends when other boundary conditions are used, global existence fails, essentially because the operator  $M_\theta$  does not have norm bounded away from 1; it is always less than 1 for continuous  $\theta$ , but can come arbitrarily close to 1, and when it does we cannot solve for  $\sigma$ . We also discuss the curvature and stability of solutions.

**3.1. The periodic case.** The periodic case is somewhat more involved since the space of unit-speed curves is not equivalent to the space of angle functions. To handle this, we assume our curves are normalized to have center of mass at the origin; this is no loss of generality since the center of mass would just move in a straight line anyway. This assumption replaces the fixed point assumption in the previous sections. We will suppose the curves are defined on the circle  $S^1$  of length  $2\pi$ . Define the ambient space of mean-zero curves by

$$\mathcal{X} = \left\{ \gamma \in C^1(S^1, \mathbb{R}^2) \mid \int_{S^1} \gamma(s) ds = 0 \right\}.$$

We want to describe the subset of unit-speed curves in terms of their angle functions, so that  $\gamma'(s) = (\cos \theta(s), \sin \theta(s))$ , but notice that not every angle function can represent a closed curve: since  $\gamma(2\pi) - \gamma(0) = 0$ , we must have

$$(32) \quad \int_0^{2\pi} \gamma'(s) ds = 0 \quad \implies \quad \int_0^{2\pi} \exp(i\theta(s)) ds = 0.$$

As long as  $\theta$  is continuous, periodic, and satisfies (32), we can reconstruct a unique mean-zero  $\gamma$  by the formula

$$(33) \quad \gamma(s) = \frac{1}{2\pi} \int_0^{2\pi} x \exp(i\theta(x)) dx + \int_0^s \exp(i\theta(x)) dx.$$

The tangent spaces to  $\mathcal{A}$  consist of angular velocity functions  $\omega$  which must preserve (32) to first order, i.e.,  $\omega \in C(S^1, \mathbb{R})$  is in  $T_\gamma \mathcal{A}$  if and only if

$$(34) \quad \int_0^{2\pi} \omega(s) \exp(i\theta(s)) ds = 0.$$

The orthogonal projection can be constructed as in Theorem 1.2; for a mean-zero vector field  $w$  in  $T_\gamma \mathcal{X}$  with  $\gamma = F(\theta)$  as in (33), the orthogonal decomposition looks exactly the same as in (15), with the only difference being that the Green function  $G$  defining the operators  $M_\theta$  and  $N_\theta$  is given by

$$(35) \quad G(s, x) = \frac{1}{2\alpha \sinh \frac{\pi}{\alpha}} \begin{cases} \cosh\left(\frac{\pi+s-x}{\alpha}\right) & 0 \leq s \leq x, \\ \cosh\left(\frac{\pi+x-s}{\alpha}\right) & x \leq s \leq 2\pi. \end{cases}$$

Thus the geodesic equation given by (17) takes the same form.

Now local existence depends only on smoothness of the operators  $M_\theta$  and  $N_\theta$  (which is still valid by the same arguments) and on the invertibility of  $1 - M_\theta$ , and this is what fails. A direct computation shows that  $\int_0^{2\pi} G(s, x) dx = 1$  for every  $s$ , which means that the norm of  $M_\theta$  is not necessarily bounded away from 1 uniformly in  $\theta$ . Rather we must actually incorporate properties of  $\theta$ . The  $L^\infty$  norm of  $M_\theta$  is

$$\|M_\theta\|_{L^\infty} = \sup_s \int_0^{2\pi} G(s, x) |\cos(\theta(x) - \theta(s))| dx;$$

by continuity the supremum is attained at some  $s_0$ , and the only way we could have  $\|M_\theta\| = 1$  is if  $\theta(x)$  is equal to either  $\theta(s_0)$  or  $\theta(s_0) + \pi$  for all  $x$ , which is impossible without  $\theta$  being constant. However, if  $\theta$  were only in  $L^\infty$  and not necessarily continuous, then the norm of  $M_\theta$  could easily be 1: for example if

$$(36) \quad \theta(x) = \begin{cases} 0 & 0 \leq x < \pi, \\ \pi & \pi \leq x < 2\pi, \end{cases}$$

then  $\theta$  satisfies (32) and  $M_\theta \cos \theta = \cos \theta$ , which means  $1 - M_\theta$  has a nontrivial kernel in  $L^\infty$  and cannot be invertible. This corresponds to a loop which goes out a distance  $\pi$  along the  $x$ -axis at unit speed and then comes back the same way. Although the corresponding  $\gamma$  is not  $C^1$ , we can easily approximate it in the  $H^1$  Riemannian distance generated by (5).

This phenomenon is a very explicit illustration of the general difficulty in working with weak Riemannian metrics on infinite-dimensional manifolds. If  $\mathcal{X}$  has the  $C^1$  topology and  $\mathcal{A}$  has the  $C^0$  topology, then

(33) embeds  $\mathcal{A}$  as a smooth Banach submanifold, just as in the one-fixed-endpoint case. If we instead used the  $W^{1,\infty}$  topology on  $\mathcal{X}$  and the  $L^\infty$  topology on  $\mathcal{A}$ , we would *not* get a submanifold. Any curve for which  $|\cos \theta|$  and  $|\sin \theta|$  are constant will be a singular point of the image of  $\mathcal{A}$  in  $\mathcal{X}$ , and the failure of the orthogonal projection at such curves is a consequence of this lack of smoothness.

This is somewhat difficult to visualize, so we will present a very simple example below to explain what is happening.

First we motivate the example by considering the evolution of discontinuities arising from piecewise continuous initial data. We find that for  $\alpha > 0$  such discontinuities cannot propagate along the string and so the domains of continuity stay fixed. The resulting piecewise differentiable position field  $\eta(s, t)$  can therefore be expected to evolve in a similar way to such piecewise linear solutions derived from an analogous finite-dimensional model, at least in having ‘corners’ located at fixed values of  $s$ .

In the following, we let  $\chi_n \subset [0, 2\pi)$  be an arbitrary set of  $n$  points at which either  $\theta_0(s)$  or  $\omega_0(s)$  has a finite jump discontinuity.

**Theorem 3.1.** *Given initial data  $\theta_0 \in C([0, 2\pi) \setminus \chi_n, \mathbb{R})$  subject to (33), with  $\omega_0 \in C([0, 2\pi) \setminus \chi_n, \mathbb{R})$  satisfying (34) for  $\theta = \theta_0$ , the only points at which solutions to system (19) may exhibit discontinuities are in the set  $\chi_n$ .*

*Proof.* We have, from equation (22),  
(37)

$$\alpha^2(i\theta_{tt} - \theta_t^2) = -\sigma + \exp(-i\theta(s, t)) \int_0^{2\pi} G(s, x)\sigma(x, t) \exp(i\theta(x, t)) dx.$$

Suppose that at some time  $t$ ,  $\eta_s(s, t)$  admits a jump discontinuity at a (fixed) reference point  $s = \mathcal{S}$ , with  $\lim_{s \rightarrow \mathcal{S}^+} \theta(s, t) = \theta_+(t)$  and  $\lim_{s \rightarrow \mathcal{S}^-} \theta(s, t) = \theta_-(t)$ . Denote by  $[\theta]_{\mathcal{S}}$  the jump,  $\theta(\mathcal{S}_+, t) - \theta(\mathcal{S}_-, t)$ . Owing to the continuity of  $G(s, x)$ , equation (37) then gives

$$(38) \quad \alpha^2(i[\theta_{tt}]_{\mathcal{S}} - [\theta_t^2]_{\mathcal{S}}) = -[\sigma]_{\mathcal{S}} + [\exp(-i\theta)]_{\mathcal{S}} \int_0^{2\pi} G(\mathcal{S}, x)\sigma(x, t) \exp(i\theta(x, t)) dx.$$

Since  $[\exp(-i\theta)]_{\mathcal{S}} = -i\sqrt{2} \operatorname{sgn}(\theta_+ - \theta_-)(1 - \cos(\vartheta_+ - \theta_-))^{1/2} \exp(-i\bar{\theta})$ , with  $\bar{\theta} = \frac{1}{2}(\theta_- + \theta_+)$ , equation (38) becomes

$$\alpha^2(i[\theta_{tt}]_{\mathcal{S}} - [\theta_t^2]_{\mathcal{S}}) = -2i \sin \frac{[\theta]_{\mathcal{S}}}{2} (\mathcal{U} + i\mathcal{V}) - [\sigma]_{\mathcal{S}}$$

where

$$\begin{aligned} \mathcal{U} &= \cos \bar{\theta}(\mathcal{S}, t) \int_0^{2\pi} G(\mathcal{S}, x) \sigma(x, t) \cos(\theta(x, t)) dx \\ &\quad + \sin \bar{\theta}(\mathcal{S}, t) \int_0^{2\pi} G(\mathcal{S}, x) \sigma(x, t) \sin(\theta(x, t)) dx, \end{aligned}$$

and

$$\begin{aligned} \mathcal{V} &= -\sin \bar{\theta}(\mathcal{S}, t) \int_0^{2\pi} G(\mathcal{S}, x) \sigma(x, t) \cos(\theta(x, t)) dx \\ &\quad + \cos \bar{\theta}(\mathcal{S}, t) \int_0^{2\pi} G(\mathcal{S}, x) \sigma(x, t) \sin(\theta(x, t)) dx. \end{aligned}$$

Since  $\mathcal{S}$  is fixed, we thereby have the jump relations

$$(39) \quad \alpha^2 \frac{d^2 [\theta]_{\mathcal{S}}}{dt^2} + 2 \sin \frac{[\theta]_{\mathcal{S}}}{2} \mathcal{U} = 0$$

and

$$(40) \quad [\sigma]_{\mathcal{S}} = \alpha^2 [\theta_t^2]_{\mathcal{S}} + 2 \sin \frac{[\theta]_{\mathcal{S}}}{2} \mathcal{V}.$$

Integrating (39) gives

$$(41) \quad [\theta]_{\mathcal{S}}(t) = [\theta_0]_{\mathcal{S}} + t[\omega_0]_{\mathcal{S}} - 2\alpha^{-2} \int_0^t (t - \eta) \sin \frac{[\theta]_{\mathcal{S}}(\eta)}{2} \mathcal{U}(\eta) d\eta$$

where  $\theta(s, 0) = \theta_0(s)$ ,  $\theta_t(s, 0) = \omega_0(s)$ . As a consequence of the existence result (bounding  $\mathcal{U}$ ), Gronwall's inequality together with the Lipschitz continuity of  $\sin \frac{[\theta]_{\mathcal{S}}(\eta)}{2}$  implies

$$(42) \quad |[\theta]_{\mathcal{S}}(t)| \leq (|[\theta_0]_{\mathcal{S}}| + t|[\omega_0]_{\mathcal{S}}|) \exp(ct^2), \quad t > 0,$$

for some generic constant  $c$ .

Clearly, if  $\theta_0(s), \omega_0(s) \in C([0, 2\pi] \setminus \chi_n)$ , then no points at which discontinuities in  $\theta(s, t)$  may occur for  $t > 0$  lie outside  $\chi_n$ .  $\square$

As in [P1], we can geometrically approximate the unit-speed curves by a finite set of points joined by rigid rods of unit length; there it was shown that the geodesic equation for the finite-dimensional configuration space gives a good approximation of the infinite-dimensional geodesic equation (with  $\alpha = 0$ ). We take a very small example: consider four points in the plane  $\eta_1, \eta_2, \eta_3, \eta_4$ , and assume that the center of mass is zero, i.e.,  $\sum_k \eta_k = 0$ . This is a six-dimensional configuration space which is the analogue of  $\mathcal{X}$ . Now impose the conditions

$$|\eta_1 - \eta_4| = |\eta_2 - \eta_1| = |\eta_3 - \eta_2| = |\eta_4 - \eta_3| = 2,$$

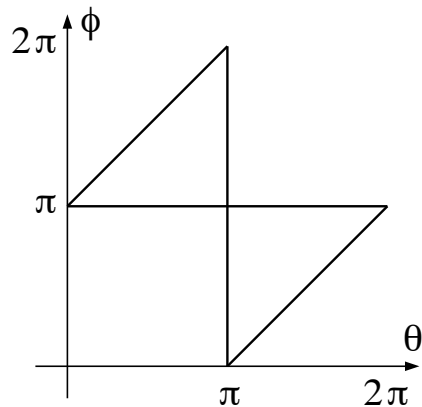


FIGURE 3. The graph of  $1 + \cos \theta + \cos \phi + \cos (\theta - \phi) = 0$ .

the rescaling by 2 is geometrically irrelevant but simplifies one formula. Roughly speaking, imposing these four conditions reduces our analogue of the space  $\mathcal{A}$  to a two-dimensional set. By possibly applying a rotation and/or a reflection of  $\mathbb{R}^2$ , it is clear that we can arrange the points so that  $\eta_2 = \eta_1 + (2, 0)$ , and then it is easy to see that the length constraints imply that

$$\begin{aligned} \eta_1 &= (a, b), & \eta_2 &= (a + 2, b), & \eta_3 &= (a + 2 + 2 \cos \theta, b + 2 \sin \theta), \\ & & & & \eta_4 &= (a - 2 \cos \phi, b - 2 \sin \phi), \end{aligned}$$

where  $a$  and  $b$  are determined by the mean-zero condition and  $\theta$  and  $\phi$  must satisfy the equation

$$(43) \quad 1 + \cos \theta + \cos \phi + \cos (\theta - \phi) = 0.$$

The solution set of this equation is shown in Figure 3. It is clearly *not* a one-dimensional manifold, having three singular points as a subset of the torus  $S^1 \times S^1$ . The horizontal and vertical lines correspond to hinges (where  $\eta_1 = \eta_3$  or  $\eta_2 = \eta_4$ ), while the diagonal lines correspond to rhombuses; the intersection points correspond to straight segments (i.e., straight hinges or degenerate rhombuses). The arc space  $\mathcal{A}$  is the two-dimensional product of this one-dimensional set with the group of Euclidean motions (i.e., two disjoint circles corresponding to rotation and reflection or simple rotation). It thus cannot be a smooth manifold.

To see what happens geometrically, we now use the analogue of the Riemannian metric (5) on  $\mathbb{R}^8$ , which we will define by the formula

$$\langle\langle \dot{\eta}, \dot{\eta} \rangle\rangle = \sum_{k=1}^4 |\dot{\eta}_k|^2 + \alpha^2 \sum_{k=1}^4 |\dot{\eta}_k - \dot{\eta}_{k+1}|^2,$$

identifying  $\eta_5 = \eta_1$  cyclically. Geodesics in this metric are straight lines, and if the center of mass is not initially moving then it never moves. We now embed  $\mathcal{A}$  in  $\mathbb{R}^8$  and see what kind of metric we get (at least on the nonsingular points, the rhombuses and hinges); as coordinates we use the rhombus angle  $\phi$  or the hinge angle (which is either  $\theta$  or  $\phi$ ; we may as well assume  $\theta$  is being used for both), and the rotation angle  $\beta$ . The Riemannian metric reduces in these coordinates to

$$\langle\langle (\dot{\theta}, \dot{\beta}), (\dot{\theta}, \dot{\beta}) \rangle\rangle = (4 + 8\alpha^2)(\dot{\theta}^2 + 2\dot{\beta}\dot{\theta} + 2\dot{\beta}^2),$$

which is *flat*; the geodesic equations are  $\ddot{\theta} = \ddot{\beta} = 0$ . Hence angular momentum is conserved, and we may as well assume there is no rotation at all (which reduces  $\mathcal{A}$  to the one-dimensional singular set depicted in Figure 3. Every geodesic corresponds to a constant-speed motion of a particle along this set, and this makes it obvious that we cannot have well-posedness, since any geodesic will eventually hit one of the three singular points. Physically such a geodesic corresponds to a rhombus or hinge for which the angle changes with constant speed until it degenerates to a straight segment, and the problem is that there is nothing to stop the geodesic from changing direction at one of the singular points (e.g., a rhombus could collapse to a segment and then start bending as a hinge, or expand again into a rhombus). See Figure 4.

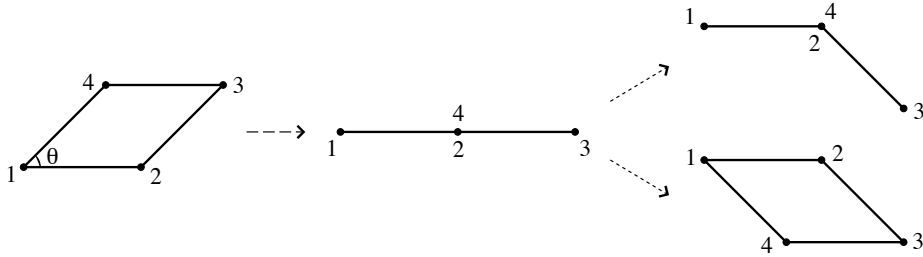


FIGURE 4. Four points rigidly connected by rods of equal length have a configuration space which is not a manifold. A rhombus can collapse to a segment, and can either continue as a rhombus or rotate into a hinge.

We would expect exactly the same thing to happen on the actual arc space  $\mathcal{A}$  in  $L^\infty$ . We can imagine a  $C^1$  approximation of a hinge or rhombus which collapses to a segment in finite time. This happens since although the  $C^1$  arc space  $\mathcal{A}$  is a smooth manifold, its geometric completion in the  $H^1$  Riemannian distance includes straight segments, which are singular points; hence the  $H^1$  arc space cannot be a smooth manifold. Thus we should be able to get blowup by finding a length-minimizing curve in the  $H^1$  arc space joining a  $C^1$  unit-speed curve to one of these singular curves (using a direct calculus of variations argument); this path will be a  $C^1$  curve for all time until the blowup time, when the  $C^1$  norm goes to infinity.

What is interesting about this phenomenon is that it gives a toy model for how the equations of ideal fluid mechanics might blow up in three dimensions. Ebin and Marsden [EM] proved that in a sufficiently smooth Sobolev topology, the space of volume-preserving diffeomorphisms is a  $C^\infty$  manifold, but Shnirelman [Shn] has shown that in the  $L^2$  Riemannian distance, the Cauchy completion is the set of all measure-preserving measurable maps, and we can imagine solutions of the ideal fluid equations blowing up in the same way by trying to approach one of these singular maps along a length-minimizing curve.

**3.2. Curvature and stability.** Finally we discuss stability from the Arnold perspective [A], using the Riemannian curvature to measure the size of linearized perturbations (Jacobi fields). Arnold originally formulated the equations of ideal fluid mechanics as a geodesic on an infinite-dimensional Riemannian manifold in order to use the sign of the curvature as a stability test: loosely speaking, negative curvature implies Lagrangian instability, while positive curvature should imply stability.

For  $L^2$  whips the curvature is known to be strictly positive in all sections [P2]; however this information is not useful for stability analysis, since the fact that the exponential map is not smooth means that we cannot use the Rauch comparison theorem even for short time to get bounds on Jacobi fields. In the present case we have a smooth exponential map, but we will show that the curvature can occasionally be negative, which means that Jacobi fields could conceivably grow exponentially in time. We will here work with the one-fixed-point boundary condition, though things are similar with periodic or other boundary conditions. For the necessary Riemannian geometry we refer to Lang [L] and do Carmo [dC]; the latter works only in finite dimensions, but the formulas we will need all generalize to weak metrics on infinite-dimensional manifolds (see e.g., Biliotti [B]).

**Theorem 3.2.** *The Riemannian curvature of the arc space  $\mathcal{A}$  at a point  $\theta$  in the plane spanned by  $\omega_1, \omega_2 \in T_\theta \mathcal{A}$  is given by*

(44)

$$\langle \langle R(\omega_1, \omega_2)\omega_2, \omega_1 \rangle \rangle_\theta = \alpha^2 \int_0^1 \omega_1^2 (1 - M_\theta)^{-1} \omega_2^2 - \omega_1 \omega_2 (1 - M_\theta)^{-1} \omega_1 \omega_2.$$

*Proof.* The easiest way to compute the sectional curvature is to use the Gauss-Codazzi formula for Riemannian submanifolds, since  $\mathcal{A}$  is a submanifold of the flat space  $\mathcal{X}$ . To do this we need to compute the second fundamental form, but we have essentially already done this by computing the orthogonal projection (15). The Gauss-Codazzi formula says [dC] that the (unnormalized) sectional curvature of the submanifold  $\mathcal{A}$  in the plane spanned by vectors  $\omega_1, \omega_2 \in T_\theta \mathcal{A}$  is

$$(45) \quad \langle \langle R(\omega_1, \omega_2)\omega_2, \omega_1 \rangle \rangle = \langle \langle B(\omega_1, \omega_1), B(\omega_2, \omega_2) \rangle \rangle - \|B(\omega_1, \omega_2)\|^2,$$

where  $B$  is the second fundamental form (a bilinear operator from  $T_\gamma \mathcal{A}$  to  $(T_\gamma \mathcal{A})^\perp$ ). We already know the second fundamental form, since the geodesic equation on any submanifold of a flat space is always given by

$$\frac{d^2}{dt^2} F(\theta) = B \left( \frac{d\theta}{dt}, \frac{d\theta}{dt} \right).$$

Since we know the geodesic equation (17), we conclude by polarization that

$$\begin{aligned} \partial_s B(\omega_1, \omega_2) &= (iN_\theta(1 - M_\theta^{-1}) - 1)\gamma' \\ &= (1 - \partial_s^2)^{-1}(h_{12}\gamma') - h_{12}\gamma', \end{aligned}$$

where  $h_{12} = (1 - M_\theta)^{-1}(\omega_1 \omega_2)$ , with  $B$  itself determined by the fixed point condition (that it is zero when  $s = 0$ ).

We therefore have

$$\langle \langle B(\omega_1, \omega_1), B(\omega_2, \omega_2) \rangle \rangle = \int_0^1 \langle \partial_s B(\omega_1, \omega_1), (\partial_s^{-2} + \alpha^2)\partial_s B(\omega_2, \omega_2) \rangle.$$

Complexifying and writing  $\partial_s B(\omega_1, \omega_1) = (f_1 + if_2)\gamma'$  and  $\partial_s B(\omega_2, \omega_2) = (g_1 + ig_2)\gamma'$  we can simplify this to

$$\langle \langle B(\omega_1, \omega_1), B(\omega_2, \omega_2) \rangle \rangle = \Re \int_0^1 (f_1 + if_2)(K_\theta + iJ_\theta + \alpha^2)(g_1 + ig_2).$$

To simplify this further, recall that we showed in formula (27) that  $(\alpha^2 + K_\theta)N_\theta = J_\theta(1 - M_\theta)$ , and by the same technique we can show that  $(\alpha^2 + K_\theta)M_\theta = K_\theta + J_\theta N_\theta$ . Using the fact that  $f_2 = -N_\theta(1 - M_\theta)^{-1}f_1$

and  $g_2 = -N_\theta(1 - M_\theta)^{-1}g_1$ , this all simplifies to

$$\begin{aligned} \langle\langle B(\omega_1, \omega_1), B(\omega_2, \omega_2) \rangle\rangle &= \alpha^2 \int_0^1 g_1(1 - M_\theta)^{-1} f_1 ds \\ &= \alpha^2 \int_0^1 \omega_2^2(1 - M_\theta)^{-1} \omega_1^2 ds, \end{aligned}$$

and similarly we get

$$\langle\langle B(\omega_1, \omega_2), B(\omega_1, \omega_2) \rangle\rangle = \alpha^2 \int_0^1 \omega_1 \omega_2 (1 - M_\theta)^{-1} \omega_1 \omega_2 ds.$$

Substituting this into (45) gives (44).  $\square$

The formula (44) is relatively simple, but it is still somewhat difficult to determine the sign in general. To make this a little easier, we will consider the simplest case of the periodic whip where the initial curve is a circle.<sup>2</sup> The curvature formula is the same as (44), with the caveat that  $\omega_1$  and  $\omega_2$  are assumed to satisfy (34). We will show that for the special case  $\theta(s) = s$ , the curvature takes both signs.

**Proposition 3.3.** *For the arc space  $\mathcal{A}$  of periodic unit-speed mean-zero curves on  $S^1$ , the curvature at the circle  $\theta(s) = s$  takes on both signs if  $\alpha$  is sufficiently large.*

*Proof.* We first compute  $M_\theta$ . Let  $h(s) = \sum_{n \in \mathbb{Z}} h_n e^{ins}$ ; then  $M_\theta h = \langle \gamma', (1 - \alpha^2 \partial_s^2)^{-1} (h \gamma') \rangle$  where  $\gamma' = (\cos s, \sin s)$ , and it is easy to compute that

$$M_\theta h(s) = \sum_{n \in \mathbb{Z}} \frac{a_n}{2} \left( \frac{1}{1 + \alpha^2(n+1)^2} + \frac{1}{1 + \alpha^2(n-1)^2} \right) e^{ins}.$$

Thus we have

$$(1 - M_\theta)^{-1} h(s) = \sum_{n \in \mathbb{Z}} \frac{[1 + \alpha^2(n+1)^2][1 + \alpha^2(n-1)^2]}{\alpha^2(n^2 + 1 + \alpha^2 n^4 - 2\alpha^2 n^2 + \alpha^2)} a_n e^{ins}.$$

This can be rewritten as  $(1 - M_\theta)^{-1} h(s) = h(s) + P_\theta h(s)$ , where  $P$  is the compact operator given by  $(Ph)(s) = \int_0^s H(s-x)h(x) dx$  for the function

$$H(s) = \frac{1}{2\pi\alpha^2} \sum_{n \in \mathbb{Z}} \frac{1 + \alpha^2(n^2 + 1)}{n^2 + 1 + \alpha^2 n^4 - 2\alpha^2 n^2 + \alpha^2} e^{ins}.$$

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<sup>2</sup>The case of a straight segment is the simplest solution when one endpoint is fixed, but at this curve it is fairly easy to see that the curvature is strictly positive, and we want to demonstrate that it can take both signs.

Since  $(1 - M_\theta)^{-1} = 1 + P_\theta$ , the curvature formula (44) obviously reduces to

$$\langle\langle R(\omega_1, \omega_2)\omega_2, \omega_1 \rangle\rangle_\theta = \int_{S^1} \omega_1^2 P_\theta(\omega_2^2) - \omega_1 \omega_2 P_\theta(\omega_1 \omega_2) ds.$$

Now  $P_\theta$  is a positive-definite operator on  $L^2$ , so the second term is always negative; thus to get negative curvature we just need to show that  $P_\theta(\omega_2^2)$  can be negative somewhere. By the operator formula it is sufficient to show that  $H(s)$  is negative somewhere. The easiest thing to do is compute  $H(\pi)$ , which is an alternating series for which the first partial sum is negative if  $\alpha > \frac{1}{\sqrt{3}}$ . Hence the full sum  $H(\pi)$  must also be negative, and thus we can set up functions  $\omega_2$  and  $\omega_1$  with small supports on opposite sides of the circle which will make both terms in the curvature formula negative.  $\square$

It seems likely that for smaller values of  $\alpha$ , there will be negative curvature along other curves as well, but the difficulty in getting precise values makes such a project beyond our scope. It suffices to note that the curvature is always strictly positive (in all sections, at all curves) for  $\alpha = 0$  (see [P2]), while for  $\alpha > 0$  it is possible to get both signs. On the other hand, smoothness of the exponential map means that the curvature is *bounded* above and below in the manifold topology, unlike the  $\alpha = 0$  case where the curvature is positive but unbounded above. A lower bound on curvature tells us (by the Rauch comparison theorem) that Jacobi fields could grow exponentially but at a rate we can estimate; unbounded curvature tells us nothing at all about growth of Jacobi fields.

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