Math 3001 Analysis 1
Homework Set 7

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Problem 1: Let $f : [a, b] \to [a, b]$ with $a < b$ be a continuous function. Prove that f has a fixed point, i.e. that there is an $x_0 \in [a, b]$ such that $f(x_0) = x_0$.
Hint: Use the Intermediate Value Theorem. (4P)

Problem 2: Prove that the function

$$f : \mathbb{R} \to \mathbb{R},\ x \mapsto \begin{cases} \exp\big(-\frac{1}{x^2}\big) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

is $\mathcal{C}^\infty$, and determine all derivatives $f^{(k)}(0)$, $k \in \mathbb{N}$.
Hint: Use Problem 4 from Homework 5. (4P)

Problem 3: Let $f : I \to \mathbb{R}$ be a function defined on an open interval I. Show that f being differentiable at $a \in I$ is equivalent to the existence of a function $E : I \to \mathbb{R}$ continuous at a such that

$$f(x) = f(a) + f'(a)(x - a) + E(x)(x - a) \quad \text{for all } x \in I$$

and $E(a) = 0$. (4P)

Problem 4: Determine the derivatives of the following functions on their maximal domains:

a) $f(x) = \dfrac{x^2 - 5x + 6}{x^2 - 3x + 2}$, b) $f(x) = \ln\big(x + \sqrt{x^2 + 1}\big)$,

c) $f(x) = \ln\big(x + \sqrt{x^2 - 1}\big)$, d) $f(x) = \sqrt{|x|^3}$.

(8P)