Cyclic homology of deformation quantizations over orbifolds

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CMS Winter 2006 Meeting
December 9-11, 2006

References

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The structure maps of a groupoid can be depicted in the diagram

$$\mathsf{G}_1 \times_{\mathsf{G}_0} \mathsf{G}_1 \xrightarrow{m} \mathsf{G}_1 \xrightarrow{i} \mathsf{G}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathsf{G}_0 \xrightarrow{u} \mathsf{G}_1,$$

where s and t are the source and target map, m is the multiplication resp. composition, i denotes the inverse and finally u the inclusion of objects by identity morphisms.

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where s and t are the source and target map, m is the multiplication resp. composition, i denotes the inverse and finally u the inclusion of objects by identity morphisms.

If the groupoid carries additionally the structure of a (not necessarily Hausdorff) smooth manifold, such that s and t are submersions, then G is called a *Lie groupoid*.

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2. For every manifold M there exists a natural groupoid structure on the cartesian product $M \times M$; one thus obtains the pair groupoid of M.
3. A proper smooth Lie group action $\Gamma \times M \to M$ gives rise to the transformation groupoid $\Gamma \ltimes M$.

Proper étale Lie groupoids and orbifolds

Definition

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Theorem

Every orbifold can be represented as the orbit space of a (Morita equivalence class of a) proper étale Lie groupoid.
(Moerdijk–Pronk)

G-sheaves and crossed product algebras

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For every G-sheaf $\mathcal{A}$ one defines the *crossed product algebra* $\mathcal{A} \rtimes G$ as the vector space $\Gamma_c(G_1, s^*\mathcal{A})$ together with the convolution product

$$[a_1 * a_2]_g = \sum_{g_1 g_2 = g} \big([a_1]_{g_1} g_2\big)[a_2]_{g_2} \text{ for } a_1, a_2 \in \Gamma_c(G_1, s^*\mathcal{A}),\ g \in G.$$

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In the following, $\mathcal{A}$ will denote the G-sheaf of smooth functions on G_0. Then $\mathcal{A} \rtimes \mathsf{G}$ is the *convolution algebra* of the groupoid G.

Tools from noncommutative geometry

Definition

A *cyclic object* in a category is a simplicial object $(X_\bullet, d, s)$ together with automorphisms (cyclic permutations) $t_k : X_k \to X_k$ satisfying the identities

$$
\begin{aligned}
d_i t_{k+1} &= \begin{cases} t_{k-1} d_{i-1} & \text{for } i \neq 0, \\ d_k & \text{for } i = 0, \end{cases} \\
s_i t_k &= \begin{cases} t_{k+1} s_{i-1} & \text{for } i \neq 0, \\ t_{k+1}^2 s_k & \text{for } i = 0, \end{cases} \\
t_k^{(k+1)} &= 1.
\end{aligned}
$$

Tools from noncommutative geometry

Definition

A *mixed complex* $(X_\bullet, b, B)$ in an abelian category is a graded object $(X_k)_{k\in\mathbb{N}}$ equipped with maps $b : X_k \to X_{k-1}$ of degree -1 and $B : X_k \to X_{k+1}$ of degree $+1$ such that $b^2 = B^2 = bB + Bb = 0$.

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Example

A cyclic object $(X_\bullet, d, s, t)$ in an abelian category gives rise to a mixed complex by putting

$$b = \sum_{i=0}^{k} (-1)^i d_i, \; N = \sum_{i=0}^{k} (-1)^{ik} t_k^i, \text{ and } B = (1 + (-1)^k t_k) s_0 N.$$

Tools from noncommutative geometry

A mixed complex gives rise to a first quadrant double complex $B_{\bullet,\bullet}(X)$

$$\begin{tikzcd}
{} \arrow[d, "b"] & {} \arrow[d, "b"] & {} \arrow[d, "b"] & {} \arrow[d, "b"] \\
X_3 \arrow[d, "b"] & X_2 \arrow[l, "B"'] \arrow[d, "b"] & X_1 \arrow[l, "B"'] \arrow[d, "b"] & X_0 \arrow[l, "B"'] \\
X_2 \arrow[d, "b"] & X_1 \arrow[l, "B"'] \arrow[d, "b"] & X_0 \arrow[l, "B"'] & \\
X_1 \arrow[d, "b"] & X_0 \arrow[l, "B"'] & & \\
X_0 & & &
\end{tikzcd}$$

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Definition

The Hochschild homology $HH_\bullet(X)$ of a mixed complex $X = (X_\bullet, b, B)$ is the homology of the $(X_\bullet, b)$-complex. The cyclic homology $HC_\bullet(X)$ is defined as the homology of the total complex associated to the double complex $B_{\bullet,\bullet}(X)$.

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- $d_i(a_0 \otimes \ldots \otimes a_k) =$

$$\begin{cases} a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_k, & \text{if } 0 \leq i \leq k-1, \\ a_k a_0 \otimes \ldots \otimes a_{k-1}, & \text{if } i = k, \end{cases}$$

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The double complex $B_{\bullet,\bullet}(A)$ associated to the mixed complex $(A^\natural_\bullet, b, B)$ is called Connes' (b, B)-complex. In this case one denotes the homologies simply by $HH_\bullet(A)$, $HC_\bullet(A)$, $HP_\bullet(A)$.

Deformation quantization

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$$\star : A[[\hbar]] \times A[[\hbar]] \to A[[\hbar]],\ (a_1, a_2) \mapsto a_1 \star a_2 = \sum_{k=0}^{\infty} \hbar^k c_k(a_1, a_2)$$

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2. One has $c_0(a_1, a_2) = a_1 \cdot a_2$ for all $a_1, a_2 \in A$.
3. For some representative $\Pi \in Z^2(A, A)$ of the Poisson structure and all $a_1, a_2 \in A$ one has

$$a_1 \star a_2 - c_0(a_1, a_2) - \frac{i}{2}\hbar\Pi(a_1, a_2) \in \hbar^2 A[[\hbar]].$$

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1. There exists a G-invariant (differential) star product on $\mathcal{A}$, the sheaf of smooth functions on G_0 (Fedosov).
2. With $\mathcal{A}^{[[\hbar]]}$ denoting the corresponding deformed G-sheaf, the crossed product algebra $\mathcal{A}^{[[\hbar]]} \rtimes \mathsf{G}$ is a deformation quantization of $\mathcal{A} \rtimes \mathsf{G}$ (Tang).

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3. The invariant algebra $\left(\mathcal{A}^{[[\hbar]]}\right)^{\mathsf{G}}$ is deformation quantization of the sheaf $\mathcal{A}^{\mathsf{G}}$ of smooth functions on the symplectic orbifold $X = \mathsf{G}_0/\mathsf{G}$ (M.P.).

Hochschild and cyclic homology of deformations of the convolution algebra

Theorem

Let G be a proper étale Lie groupoid representing a symplectic orbifold X of dimension $2n$. Then the Hochschild homology of the deformed convolution algebra $\mathbb{A}^{((\hbar))} \rtimes \mathsf{G}$ is given by

$$HH_\bullet(\mathbb{A}^{((\hbar))} \rtimes \mathsf{G}) \cong H^{2n-\bullet}_{\mathrm{orb,c}}(X, \mathbb{C}((\hbar)))\,,$$

and the cyclic homology of $\mathbb{A}^{((\hbar))} \rtimes \mathsf{G}$ by

$$HC_\bullet(\mathbb{A}^{((\hbar))} \rtimes \mathsf{G}) = \bigoplus_{k \geq 0} H^{2n+2k-\bullet}_{\mathrm{orb,c}}(X, \mathbb{C}((\hbar))).$$

(Neumaier–Pflaum–Posthuma–Tang)

Hochschild and cyclic cohomology of deformations of the convolution algebra

Theorem

The Hochschild and cyclic cohomology of $\mathbb{A}^\hbar \rtimes \mathsf{G}$ are given by

$$HH^\bullet(\mathbb{A}^{((\hbar))} \rtimes \mathsf{G}) \cong H^\bullet_{\mathrm{orb}}(X, \mathbb{C}((\hbar))),$$
$$HC^\bullet(\mathbb{A}^{((\hbar))} \rtimes \mathsf{G}) \cong \bigoplus_{k \geq 0} H^{\bullet-2k}_{\mathrm{orb}}(X, \mathbb{C}((\hbar))).$$

Furthermore, the pairing between homology and cohomology is given by Poincaré duality for orbifolds.
(Neumaier–Pflaum–Posthuma–Tang)

The algebraic index theorem for orbifolds

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Let G be a proper étale Lie groupoid representing a symplectic orbifold X. Let E and F be G-vector bundles which are isomorphic outside a compact subset of X.

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Let G *be a proper étale Lie groupoid representing a symplectic orbifold* X*. Let* E *and* F *be* G*-vector bundles which are isomorphic outside a compact subset of* X*. Then the following formula holds for the index of* $[E]-[F]$*:*

$$\mathrm{Tr}_*([E]-[F]) =$$

$$= \int_{\tilde{X}} \frac{1}{m} \frac{\mathrm{Ch}_\theta\left(\frac{R^E}{2\pi i} - \frac{R^F}{2\pi i}\right)}{\det\left(1-\theta^{-1}\exp\left(-\frac{R^\perp}{2\pi i}\right)\right)} \hat{A}\Big(\frac{R^T}{2\pi i}\Big) \exp\Big(-\frac{\iota^*\Omega}{2\pi i\hbar}\Big).$$

(Pflaum–Posthuma–Tang)

The Kawasaki index theorem

As a consequence of the algebraic index theorem for orbifolds one obtains

Theorem

Given an elliptic operator D on a reduced compact orbifold X, one has

$$\mathrm{index}(D) = \int_{\widetilde{T^*X}} \frac{1}{m} \frac{\mathrm{Ch}_\theta\left(\frac{\sigma(D)}{2\pi i}\right)}{\det\left(1 - \theta^{-1}\exp\left(-\frac{R^\perp}{2\pi i}\right)\right)} \hat{A}\Big(\frac{R^T}{2\pi i}\Big),$$

where $\sigma(D)$ is the symbol of D.
(Kawasaki)