

# Short Notes in Mathematics

Fundamental concepts every student of Mathematics should know

MARKUS J. PFLAUM

September 19, 2017

## Contents

<b>Titlepage</b>	<b>1</b>
<b>1 Foundations</b>	<b>2</b>
Set theory . . . . .	2
Cartesian products . . . . .	4
Relations . . . . .	4
Functions . . . . .	5
<b>2 Number Systems</b>	<b>8</b>
Natural numbers . . . . .	8
Real numbers . . . . .	9
<b>3 Analysis</b>	<b>10</b>
3.1 Limits and Continuity . . . . .	10
3.7 Differentiability . . . . .	10
3.13 Local and Global Extrema . . . . .	12
<b>Licensing</b>	<b>13</b>
CC BY 4.0 . . . . .	13

# 1 Foundations

## Set theory

**1.1 Naive Definition of Sets (Georg Cantor 1882)** A set is a gathering together into a whole of definite, distinct objects of our perception or of our thought which are called elements of the set.

**1.2** In modern Mathematics, set theory is developed axiomatically. Axiomatic set theory goes back to Zermelo-Fraenkel and forms the basis of modern mathematics, together with formal logic. We follow the axiomatic approach here too, even though we will not introduce all of the set theory axioms of Zermelo-Fraenkel and will not go into the subtleties of set theory.

**1.3 Axiomatic Definition of Sets (cf. Zermelo-Fraenkel 1908, 1973)** Sets are denoted by letters of the roman and greek alphabets :  $a, b, c, \dots, x, y, z, A, B, C, \dots, X, Y, Z, \alpha, \beta, \gamma, \delta, \dots, \chi, \psi, \omega$ , and  $A, B, \Gamma, \Delta, \dots, X, \Psi, \Omega$ . The fundamental relational symbols of set theory are the *equality sign*  $=$  and the *element sign*  $\in$ . Further symbols of set theory are the numerical constants  $0, 1, 2, \dots, 9$  and the symbol for the empty set  $\emptyset$ .

(S1) (Extensionality Axiom)

Two sets  $M$  and  $N$  are equal if and only if they have the same elements.

Formally:  $\forall M \forall N ((M = N) \Leftrightarrow (\forall x (x \in M) \Leftrightarrow (x \in N)))$  .

(Definition of the subset relation)

A set  $M$  is called a *subset* of a set  $N$ , in signs  $M \subset N$ , if every element of  $M$  is an element of  $N$ .

Formally:  $(M \subset N) \Leftrightarrow (\forall x (x \in M) \Rightarrow (x \in N))$  .

The extensionality axiom can now be expressed equivalently as follows:

$\forall M \forall N ((M = N) \Leftrightarrow ((M \subset N) \& (N \subset M)))$  .

(S2) (Axiom of Empty Set)

There exists a set which has no elements.

Formally:  $\exists E \forall x \neg(x \in E)$  .

The set having no elements is uniquely determined by the extensionality axiom. It is denoted by  $\emptyset$ .

(S3) (Axiom of Pairings)

For all sets  $x$  and  $y$  there exists a set containing exactly  $x$  and  $y$ .

Formally:  $\forall x \forall y \exists M \forall z (z \in M \Leftrightarrow (z = x \vee z = y))$  .

The set containing  $x$  and  $y$  as elements (and no others) is denoted  $\{x, y\}$ . If  $x = y$ , one writes  $\{x\}$  for this set.

(S4) (Separation Scheme)

Let  $M$  be a set and  $P(x)$  a formula (or in other words a property). Then there exists a set  $N$  whose elements consist of all  $x \in M$  such that  $P(x)$  holds true.

In signs:  $N = \{x \in M \mid P(x)\}$  .

Using the separation scheme one can define the intersection of two sets  $M$  and  $N$  as  $M \cap N = \{x \in M \mid x \in N\}$  and the complement  $M \setminus N$  as the set  $\{x \in M \mid x \notin N\}$ .

(S5) (Axiom of the Union)

Given two sets  $M$  and  $N$  there exists a set consisting of all elements of  $M$  and  $N$  (and no others).

Formally:  $\forall M \forall N \exists U \forall x ((x \in U) \Leftrightarrow (x \in M \vee x \in N))$ .

The set  $U$  in this formula is uniquely defined by the extensionality axiom and is called the *union of  $M$  and  $N$* . It is denoted  $M \cup N$ .

More generally, if  $M$  is a set, then there exists a set denoted by  $\bigcup M$  consisting of all elements of elements of  $M$ .

Formally:  $\forall M \exists U \forall x ((x \in U) \Leftrightarrow (\exists X (X \in M \& x \in X)))$ .

The set  $U$  in this formula then is uniquely determined and abbreviated  $\bigcup M$ .

(S6) (Power Set Axiom)

For each set  $M$  there exists a set  $\mathcal{P}(M)$  containing all subsets of  $M$  (and no others).

Formally:  $\forall M \exists P \forall x ((x \subset M) \Leftrightarrow (x \in P))$ .

The set  $P$  in this formula is uniquely defined by the extensionality axiom and is called the *power set of  $M$* . It is denoted  $\mathcal{P}(M)$ .

(S7) (Axiom of Infinity) There exists a set which contains  $\emptyset$  and with each element  $x$  also the union  $x \cup \{x\}$ .

Formally:  $\exists M (\emptyset \in M \& \forall x (x \in M \Rightarrow x \cup \{x\} \in M))$  .

(S8) (Axiom of Choice) For any set  $M$  of nonempty sets, there exists a choice function  $f$  defined on  $M$  that is a map  $M \rightarrow \bigcup M$  such that  $f(x) \in x$  for all  $x \in M$ .

Formally:  $\forall M (\emptyset \notin M \Rightarrow \exists f : M \rightarrow \bigcup M, \forall x \in M (f(x) \in x))$  .

**1.4 Remark** In the formulation of the Axiom of Choice we used the notion of a function introduced below in Definition 1.13.

**1.5 Proposition** *Let  $L, M, N$  be sets. Then the following statements hold true.*

(a) (commutativity)

$M \cup N = N \cup M$  and  $M \cap N = N \cap M$ .

(b) (associativity)

$M \cup (N \cup L) = (N \cup M) \cup L$  and  $M \cap (N \cap L) = (N \cap M) \cap L$ .

(c) (distributivity)

$M \cup (N \cap L) = (M \cup N) \cap (M \cup L)$  and  $M \cap (N \cup L) = (M \cap N) \cup (M \cap L)$ .

(d)  $M \cap N \subset M$  and  $M \subset M \cup N$ .

(e) *If for a set  $X$  the relations  $X \subset M$  and  $X \subset N$  hold true, then  $X \subset M \cap N$ . If for a set  $Y$  the relations  $M \subset Y$  and  $N \subset Y$  are satisfied, then  $M \cup N \subset Y$ .*

(f)  $M \setminus \emptyset = M$  and  $M \setminus M = \emptyset$ .

(g)  $M \setminus (M \cap N) = M \setminus N$  and  $M \setminus (M \setminus N) = M \cap N$ .

(h) (de Morgan's laws)

$$M \setminus (N \cup L) = (M \setminus N) \cap (M \setminus L) \text{ and } M \setminus (N \cap L) = (M \setminus N) \cup (M \setminus L).$$

### Cartesian products

**1.6 Definition** Let  $X$  and  $Y$  be sets. For all  $x \in X$  and  $y \in Y$  the *pair*  $(x, y)$  is defined as the set  $\{\{x\}, \{x, y\}\}$ . The *cartesian product*  $X \times Y$  is defined as

$$\{z \in \mathcal{P}(\mathcal{P}(X \cup Y)) \mid \exists x \in X \exists y \in Y : z = \{\{x\}, \{x, y\}\}\}.$$

**1.7 Proposition** For sets  $X, Y$  and elements  $x, x' \in X$  and  $y, y' \in Y$  the pairs  $(x, y)$  and  $(x', y')$  are equal if and only if  $x = x'$  and  $y = y'$ .

**1.8 Proposition** Let  $L, M, N$  be sets. Then the following distributivity laws hold true:

- (a)  $L \times (M \cup N) = (L \times M) \cup (L \times N)$  and  $(M \cup N) \times L = (M \times L) \cup (N \times L)$ .
- (b)  $L \times (M \cap N) = (L \times M) \cap (L \times N)$  and  $(M \cap N) \times L = (M \times L) \cap (N \times L)$ .
- (c)  $L \times (M \setminus N) = (L \times M) \setminus (L \times N)$  and  $(M \setminus N) \times L = (M \times L) \setminus (N \times L)$ .

### Relations

**1.9 Definition** A triple  $R = (X, Y, \Gamma)$  with  $X$  and  $Y$  being sets and  $\Gamma$  a subset of the cartesian product  $X \times Y$  is called a *relation from  $X$  to  $Y$* . If  $Y = X$ , a relation  $(X, X, \Gamma)$  is called a *relation on  $X$* . The set  $\Gamma$  is called the graph of the relation.

If  $(x, y) \in \Gamma$ , one says that  $x$  and  $y$  are in relation, and denotes that by  $x R y$ .

**1.10 Remark** Usually, a relation is often denoted by a symbol like for example  $\sim$ ,  $\leq$  or  $\equiv$ . The statement that  $x$  and  $y$  are in relation is then symbolically written  $x \sim y$ ,  $x \leq y$ ,  $x \equiv y$ , respectively.

**1.11 Definition** A relation  $\sim$  on a set  $X$  is called an *equivalence relation* if it has the following properties:

- (E1) Reflexivity  
For all  $x \in X$  the relation  $x \sim x$  holds true.
- (E2) Symmetry  
For all  $x, y \in X$ , if  $x \sim y$  holds true, then  $y \sim x$  is true as well.
- (E3) Transitivity  
For all  $x, y, z \in X$  the relations  $x \leq y$  and  $y \leq x$  entail  $x \leq z$ .

**1.12 Definition** A set  $X$  together with a relation  $\leq$  on it is called an *ordered set*, a *partially ordered set* or a *poset* if the following axioms are satisfied:

- (O1) Reflexivity  
For all  $x \in X$  the relation  $x \leq x$  holds true.

(O2) Antisymmetry

If  $x \leq y$  and  $y \leq x$  for some  $x, y \in X$ , then  $x = y$ .

(O3) Transitivity

For all  $x, y, z \in X$  the relations  $x \leq y$  and  $y \leq x$  entail  $x \leq z$ .

The relation  $\leq$  is then called an *order relation* or an *order* on  $X$ .

If in addition Axiom (O4) below holds true,  $(X, \leq)$  is called a *totally ordered set* and  $\leq$  a *total order* on  $X$ .

(O4) Totality

For all  $x, y \in X$  the relation  $x \leq y$  or the relation  $y \leq x$  holds true.

A total order relation  $\leq$  on  $X$  satisfies the following law of trichotomy, where  $x < y$  stands for  $x \leq y$  and  $x \neq y$ :

Law of Trichotomy

For all  $x, y \in X$  exactly one of the statements  $x < y$ ,  $x = y$  or  $y < x$  holds true.

## Functions

**1.13 Definition** By a *function*  $f$  one understands a triple  $(X, Y, \Gamma)$  consisting of

- (a) a set  $X$ , called the *domain* of the function,
- (b) a set  $Y$ , called the *range* or *target* of the function,
- (c) a set  $\Gamma$  of pairs  $(x, y)$  of points  $x \in X$  and  $y \in Y$ , called the *graph* of the function, such that for each  $x \in X$  there is a unique  $f(x) \in Y$  with  $(x, f(x)) \in \Gamma$ .

A function  $f$  with domain  $X$ , range  $Y$  and graph  $\Gamma = \{(x, y) \in X \times Y \mid y = f(x)\}$  will be denoted

$$f : X \rightarrow Y, x \mapsto f(x).$$

**1.14 Example** The following are examples of functions:

- (a) the identity function on a set  $X$ ,  $\text{id}_X : X \rightarrow X, x \mapsto x$ ,
- (b) polynomial functions which are functions of the form  $p : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto \sum_{k=0}^n a_k x^k$ , where the  $a_k, k = 0, \dots, n$  are real numbers called the coefficients of the polynomial function,
- (c) the absolute value function  $|\cdot| : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto |x| = \sqrt{x^2}$ ,  
the euclidean norm  $\|\cdot\| : \mathbb{R}^2 \rightarrow \mathbb{R}, (x, y) \mapsto \sqrt{x^2 + y^2}$  on  $\mathbb{R}^2$  and more generally the euclidean norm  $\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}, (x_1, \dots, x_n) \mapsto \sqrt{\sum_{i=1}^n x_i^2}$  on  $\mathbb{R}^n$ .

Further examples of functions defined on (subsets of)  $\mathbb{R}$  are the exponential function, the logarithm, the trigonometric functions, and so on. Precise definitions of these will be introduced later.

**1.15 Definition** Functions of the form  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  such that

(L1)  $f(0) = 0$  and

(L2)  $f(v + w) = f(v) + f(w)$  for all  $v, w \in \mathbb{R}^m$

are called *linear*. A function  $g : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is called *affine* if there exists an  $a \in \mathbb{R}^n$  and a linear function  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  such that  $g(v) = f(v) + a$  for all  $v \in \mathbb{R}^m$ .

**1.16 Definition** A function  $f : X \rightarrow Y$  is called

- *injective* or *one-to-one* if for all  $x_1, x_2 \in X$  the equality  $f(x_1) = f(x_2)$  implies  $x_1 = x_2$ ,
- *surjective* or *onto* if for all  $y \in Y$  there exists an  $x \in X$  such that  $f(x) = y$ , and
- *bijective* if it is injective and surjective.

**1.17 Definition** Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be two functions. The *composition*  $g \circ f : X \rightarrow Z$  then is defined as the function with domain  $X$ , range  $Z$  and graph  $\Gamma = \{(x, z) \in X \times Z \mid z = g(f(x))\}$ . In other words,  $g \circ f$  maps an element  $x \in X$  to  $g(f(x)) \in Z$ .

**1.18 Definition** A function  $f : X \rightarrow Y$  is called *invertible* if there exist a function  $g : Y \rightarrow X$  such that  $g \circ f = \text{id}_X$  and  $f \circ g = \text{id}_Y$ .

**1.19 Theorem** A function  $f : X \rightarrow Y$  is invertible if and only if it is bijective.

**1.20 Definition** Let  $f : X \rightarrow Y$  be a function. For every subset  $A \subset X$  one defines the *image* of  $A$  under  $f$  as the set

$$f(A) = \{f(x) \in Y \mid x \in A\} .$$

The set  $f(X)$  is called the *image* of the function  $f$  and is usually denoted by the symbol  $\text{im}(f)$ . In case  $B$  is a subset of  $Y$ , the *preimage* of  $B$  under  $f$  is defined as the set

$$f^{-1}(B) \subset X .$$

**1.21 Remarks** (a) The notation  $f^{-1}$  in the preceding definition does not mean that  $f$  is invertible. The preimage map  $f^{-1}$  associated to a function  $f : X \rightarrow Y$  has domain  $\mathcal{P}(Y)$  and range  $\mathcal{P}(X)$ , whereas the inverse map  $g^{-1}$  of an invertible function  $g : X \rightarrow Y$  has domain  $Y$  and range  $X$ . The context will always make clear which of these two functions is meant when using the “to the power negative one” symbol on functions. If  $f : X \rightarrow Y$  is invertible, the preimage map  $f^{-1} : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$  and the inverse map  $f^{-1} : Y \rightarrow X$  are related by

$$f^{-1}(\{y\}) = \{f^{-1}(y)\} \quad \text{for all } y \in Y .$$

(b) The image  $\text{im}(f)$  of a function  $f : X \rightarrow Y$  is always contained in the range  $Y$ . Equality  $\text{im}(f) = Y$  holds if and only if  $f$  is surjective.

**1.22 Proposition** For a function  $f : X \rightarrow Y$  between two sets  $X$  and  $Y$  the following statements hold true for the images respectively preimages under  $f$  of subsets  $A, A_i, E \subset X$  with  $i \in I$  and  $B, B_j, F \subset Y$  with  $j \in J$ :

(a) *The relations*

$$f^{-1}(f(A)) \supset A \quad \text{and} \quad f(f^{-1}(B)) = B \cap \text{im}(f) \subset B$$

*hold true.*

(b) *The image map  $f : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$  preserves unions that is*

$$f\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} f(A_i) .$$

(c) *In regard to intersections, the image map  $f : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$  fulfills the relation*

$$f\left(\bigcap_{i \in I} A_i\right) \subset \bigcap_{i \in I} f(A_i) .$$

(d) *With respect to set-theoretic complement the image map  $f : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$  acts as follows:*

$$f(A) \setminus f(E) \subset f(A \setminus E) \quad \text{if } E \subset A .$$

(e) *The preimage map  $f^{-1} : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$  preserves unions that is*

$$f^{-1}\left(\bigcup_{j \in J} B_j\right) = \bigcup_{j \in J} f^{-1}(B_j) .$$

(f) *The preimage map  $f^{-1} : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$  preserves intersections that is*

$$f^{-1}\left(\bigcap_{j \in J} B_j\right) = \bigcap_{j \in J} f^{-1}(B_j) .$$

(g) *The preimage map  $f^{-1} : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$  acts as follows with respect to set-theoretic complement:*

$$f^{-1}(B) \setminus f^{-1}(F) = f^{-1}(B \setminus F) \quad \text{if } F \subset B .$$

**1.23 Remark** The image map  $f : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$  does in general not preserve intersections. To see this consider the square function  $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x^2$ . Then

$$f((-\infty, 0] \cap [0, \infty)) = \{0\} \quad \text{but} \quad f((-\infty, 0]) \cap f([0, \infty)) = [0, \infty) .$$

If  $f$  is an injective map, the corresponding preimage map preserves all intersections. Actually this is even a sufficient criterion for injectivity of  $f$ .

## 2 Number Systems

### Natural numbers

**2.1 Definition (Peano)** A triple  $(\mathbb{N}, 0, s)$  consisting of a set  $\mathbb{N}$ , an element  $0 \in \mathbb{N}$  called *zero element* and a map  $s : \mathbb{N} \rightarrow \mathbb{N}$  called *successor map* is called a *system of natural numbers* if the following axioms hold true:

- (P1) 0 is not in the image of  $s$ .
- (P2)  $s$  is injective.
- (P3) (Induction Axiom) Every inductive subset of  $\mathbb{N}$  coincides with  $\mathbb{N}$ , where by an *inductive subset of  $\mathbb{N}$*  one understands a set  $I \subset \mathbb{N}$  having the following properties:
  - (I1) 0 is an element of  $I$ .
  - (I2) If  $n \in I$ , then  $s(n) \in I$ .

By Axiom (P1), 0 is not in the image of the successor map. But all other elements of the Peano structure are.

**2.2 Proposition** *Let  $(\mathbb{N}, 0, s)$  be a system of natural numbers. Then the image of  $s$  coincides with the set  $\mathbb{N}_{\neq 0} := \{n \in \mathbb{N} \mid n \neq 0\}$  of all non-zero elements, in signs  $s(\mathbb{N}) = \mathbb{N}_{\neq 0}$ .*

**2.3 Theorem** *The set  $\mathbb{N}$  of natural numbers together with addition  $+$  :  $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ , multiplication  $\cdot$  :  $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  and the elements 0 and  $1 := s(0)$  satisfies the following axioms:*

- (A1) (associativity)  
 $l + (m + n) = (l + m) + n$  for all  $l, m, n \in \mathbb{N}$ .
- (A2) (commutativity)  
 $m + n = n + m$  for all  $m, n \in \mathbb{N}$ .
- (A3) (neutrality of 0)  
 $m + 0 = 0 + m = m$  for all  $m \in \mathbb{N}$ .
- (M1) (associativity)  
 $l \cdot (m \cdot n) = (l \cdot m) \cdot n$  for all  $l, m, n \in \mathbb{N}$ .
- (M2) (commutativity)  
 $m \cdot n = n \cdot m$  for all  $m, n \in \mathbb{N}$ .
- (M3) (neutrality of 1)  
 $m \cdot 1 = 1 \cdot m = m$  for all  $m \in \mathbb{N}$ .
- (D) (distributivity)  
 $l \cdot (m + n) = (l \cdot m) + (l \cdot n)$  and  
 $(m + n) \cdot l = (m \cdot l) + (n \cdot l)$  for all  $l, m, n \in \mathbb{N}$ .

*In other words,  $\mathbb{N}$  together with  $+$  and  $\cdot$  and the elements 0, 1 is a semiring.*



**Real numbers**

**2.4 Definition** By a *field of real numbers* one understands a set  $\mathbb{R}$  together together with operations  $+$  :  $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  and  $\cdot$  :  $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  called *addition* and *multiplication*, two distinct elements 0 and 1 and an order relation  $\leq$  such that the following axioms are satisfied:

(A1) (associativity)

$$x + (y + z) = (x + y) + z \text{ for all } x, y, z \in \mathbb{R}.$$

(A2) (commutativity)

$$x + y = y + x \text{ for all } x, y \in \mathbb{R}.$$

(A3) (neutrality of 0)

$$x + 0 = 0 + x = x \text{ for all } x \in \mathbb{R}.$$

(A4) (additive inverses)

For every  $x \in \mathbb{R}$  there exists  $y \in \mathbb{R}$ , called *negative* of  $x$  such that  $x + y = y + x = 0$ . The negative of  $x$  is usually denoted  $-x$ .

(M1) (associativity)

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \text{ for all } x, y, z \in \mathbb{R}.$$

(M2) (commutativity)

$$x \cdot y = y \cdot x \text{ for all } x, y \in \mathbb{R}.$$

(M3) (neutrality of 1)

$$x \cdot 1 = 1 \cdot x = x \text{ for all } x \in \mathbb{R}.$$

(M4) (multiplicative inverses of nonzero elements)

For every  $x \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$  there exists  $y \in \mathbb{R}$ , called *inverse* of  $x$  such that  $x \cdot y = y \cdot x = 1$ . The inverse of  $x \neq 0$  is usually denoted  $x^{-1}$ .

(D) (distributivity)

$$\begin{aligned} x \cdot (y + z) &= (x \cdot y) + (x \cdot z) && \text{and} \\ (x + y) \cdot z &= (x \cdot z) + (y \cdot z) && \text{for all } x, y, z \in \mathbb{R}. \end{aligned}$$

(O5) (monotony of addition)

For all  $x, y, z \in \mathbb{R}$  the relation  $x \leq y$  implies  $x + z \leq y + z$ .

(O6) (monotony of multiplication)

For all  $x, y, z \in \mathbb{R}$  with  $z \geq 0$  the relation  $x \leq y$  implies  $x \cdot z \leq y \cdot z$ .

(C) (completeness)

Each non-empty subset  $X \subset \mathbb{R}$  bounded above has a least upper bound.

In other words,  $\mathbb{R}$  together with  $+$  and  $\cdot$ , the elements 0, 1 and the order relation  $\leq$  is a complete ordered field.

**2.5 Theorem** *There exists (up to isomorphism) only one field of real numbers  $\mathbb{R}$ .*

### 3 Analysis

#### 3.1 Limits and Continuity

**3.2 Definition** Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence of real numbers. The sequence is said to *converge* to a real number  $x \in \mathbb{R}$  if for every  $\varepsilon > 0$  there exists an  $N \in \mathbb{N}$  such that

$$|x_n - x| < \varepsilon \text{ for all natural } n \geq N .$$

One then calls  $(x_n)_{n \in \mathbb{N}}$  a *convergent* sequence and  $x$  its *limit*.

**3.3 Proposition** *The limit of a convergent sequence is uniquely determined.*

**3.4 Definition** The function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  has the *limit*  $L$  at the point  $(a, b) \in \mathbb{R}^2$ , written

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L,$$

if for every  $\varepsilon > 0$  there is a  $\delta > 0$  such that

$$|f(x, y) - L| < \varepsilon \text{ for all } (x, y) \neq (a, b) \text{ with } d((x, y), (a, b)) < \delta.$$

**3.5 Remark** Intuitively,  $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$  means that  $f(x, y)$  is as close to  $L$  as we wish whenever the distance of the point  $(x, y)$  to  $(a, b)$  is sufficiently small.

**3.6 Definition** The function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is called *continuous* at the point  $(a, b) \in \mathbb{R}^2$ , if

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b).$$

The function  $f$  is said to be *continuous* on a region  $R \subset \mathbb{R}^2$ , if it is continuous at every point  $(a, b) \in R$ .

#### 3.7 Differentiability

**3.8 Definition** A function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $(x, y) \mapsto f(x, y)$  is called *partially differentiable* in the point  $(a, b) \in \mathbb{R}^2$  with respect to the variable  $x$  (resp.  $y$ ), if the limit

$$\left( \begin{array}{l} \frac{\partial f}{\partial x}(a, b) := \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h} \\ \text{resp. } \frac{\partial f}{\partial y}(a, b) := \lim_{h \rightarrow 0} \frac{f(a, b+h) - f(a, b)}{h} \end{array} \right)$$

exists. One then calls  $\frac{\partial f}{\partial x}(a, b)$  and  $\frac{\partial f}{\partial y}(a, b)$  the *partial derivatives* of  $f$  at  $(a, b)$ . If  $f$  is partially differentiable in every point of  $\mathbb{R}^2$  with respect to the variables  $x$  and  $y$ , then one says that  $f$  is *partially differentiable*.

A function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $(x, y) \mapsto f(x, y)$  is called *twice partially differentiable*, if it is partially differentiable, and if the partial derivatives  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$  are partially differentiable as well.

A function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $(x, y) \mapsto f(x, y)$  is called *differentiable in the point*  $(a, b) \in \mathbb{R}^2$ , if there exists a linear function  $L : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that

$$\lim_{(x,y) \rightarrow (a,b)} \frac{E(x, y)}{\sqrt{(x-a)^2 + (y-b)^2}} = 0,$$

where  $E : \mathbb{R}^2 \rightarrow \mathbb{R}$  is the *error function* defined by  $E(x, y) := f(x, y) - f(a, b) - L(x, y)$ . One then calls  $L$  the *linear approximation* of  $f$  at  $(a, b)$ , and writes

$$f(x, y) \approx f(a, b) + L(x, y).$$

**3.9 Definition** A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $x \mapsto f(x)$  is called *partially differentiable in the point*  $a = (a_1, \dots, a_n) \in \mathbb{R}^n$  with respect to the variable  $x_i$ , if for every  $i$ ,  $1 \leq i \leq n$  the limit

$$\frac{\partial f}{\partial x_i}(a) := \lim_{h \rightarrow 0} \frac{f(a_1, \dots, a_i + h, \dots, a_n) - f(a_1, \dots, a_n)}{h}$$

exists. One then calls  $\frac{\partial f}{\partial x_i}(a)$  the *(i-th) partial derivative* of  $f$  at  $a$  with respect to the variable  $x_i$ . If  $f$  is partially differentiable in every point of  $\mathbb{R}^n$  with respect to the variables  $x_1, \dots, x_n$ , then one says that  $f$  is *partially differentiable*.

A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $x \mapsto f(x)$  is called *twice partially differentiable*, if it is partially differentiable, and if the partial derivatives  $\frac{\partial f}{\partial x_i}$ ,  $1 \leq i \leq n$  are partially differentiable as well.

A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $x \mapsto f(x)$  is called *differentiable in the point*  $a \in \mathbb{R}^n$ , if there exists a linear function  $L : \mathbb{R}^n \rightarrow \mathbb{R}$  such that

$$\lim_{x \rightarrow a} \frac{E(x)}{\sqrt{(x_1 - a_1)^2 + \dots + (x_n - a_n)^2}} = 0,$$

where  $E : \mathbb{R}^n \rightarrow \mathbb{R}$  is the *error function* defined by  $E(x) := f(x) - f(a) - L(x)$ . One then calls  $L$  the *linear approximation* of  $f$  at  $a$ , and writes

$$f(x) \approx f(a) + L(x).$$

**3.10 Theorem** If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $x \mapsto f(x)$  is differentiable at  $a \in \mathbb{R}^n$ , then  $f$  is partially differentiable and continuous at  $a$ .

**3.11 Theorem** If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $x \mapsto f(x)$  is twice partially differentiable, and the second partial derivatives

$$\frac{\partial^2 f}{\partial x_i \partial x_j} := \frac{\partial}{\partial x_i} \frac{\partial f}{\partial x_j}$$

are continuous, then

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

for  $1 \leq i, j \leq n$ .

**3.12 Definition** If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $x \mapsto f(x)$  is a partially differentiable function, and  $x \in \mathbb{R}^n$ , the vector

$$\text{grad } f(x) := \left( \frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x) \right)$$

is called the *gradient* of  $f$  at  $x$ .

Given a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $x \mapsto f(x)$  which is partially differentiable (up to isolated points), a point  $x \in \mathbb{R}^n$  is called a *critical point* of  $f$ , if  $\text{grad } f(x) = 0$ , or if  $\text{grad } f(x)$  is not defined.

### 3.13 Local and Global Extrema

**3.14 Definition** If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $x \mapsto f(x)$  is a function, a point  $a \in \mathbb{R}^n$  is called a *local maximum* (resp. *local minimum*) of  $f$ , if  $f(x) \leq f(a)$  (resp.  $f(x) \geq f(a)$ ) for all  $x \in \mathbb{R}^n$  near  $a$ .

The point  $a \in \mathbb{R}^n$  is called a *global maximum* (resp. *global minimum*) of  $f$  over the region  $R \subset \mathbb{R}^n$ , if  $f(x) \leq f(a)$  (resp.  $f(x) \geq f(a)$ ) for all  $x \in R$ .

**3.15 Theorem** Assume that  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $(x, y) \mapsto f(x, y)$  is a twice continuously partially differentiable function. Suppose that  $(a, b)$  is a point where  $\text{grad } f(a, b) = 0$ . Let

$$D = \frac{\partial^2 f}{\partial x^2}(a, b) \cdot \frac{\partial^2 f}{\partial y^2}(a, b) - \left( \frac{\partial^2 f}{\partial x \partial y}(a, b) \right)^2.$$

- If  $D > 0$  and  $\frac{\partial^2 f}{\partial x^2}(a, b) > 0$ , then  $f$  has a local minimum in  $a$ .
- If  $D > 0$  and  $\frac{\partial^2 f}{\partial x^2}(a, b) < 0$ , then  $f$  has a local maximum in  $a$ .
- If  $D < 0$ , then  $f$  has a saddle point in  $a$ .
- If  $D = 0$ , no conclusion can be made:  $f$  can have a local maximum, a local minimum, a saddle point, or none of these in the point  $(a, b)$ .

**3.16 Definition** If  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $(x, y) \mapsto f(x, y)$  and  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $(x, y) \mapsto g(x, y)$  are functions, and  $c \in \mathbb{R}$  a number, a point  $(a, b) \in \mathbb{R}^2$  is called a *local maximum* (resp. *local minimum*) of  $f$  under the constraint  $g(x, y) = c$ , if  $f(x, y) \leq f(a, b)$  (resp.  $f(x, y) \geq f(a, b)$ ) for all  $(x, y) \in \mathbb{R}^2$  near  $(a, b)$  which satisfy  $g(x, y) = c$ .

The point  $(a, b) \in \mathbb{R}^2$  is called a *global maximum* (resp. *global minimum*) of  $f$  under the constraint  $g(x, y) = c$ , if  $f(x, y) \leq f(a, b)$  (resp.  $f(x, y) \geq f(a, b)$ ) for all  $(x, y) \in \mathbb{R}^2$  which satisfy  $g(x, y) = c$ .

**3.17 Theorem** Assume that  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $x \mapsto f(x)$  is a smooth function, and  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $x \mapsto g(x)$  a smooth constraint. If  $f$  has a maximum or minimum at the point  $(a, b)$  under the constraint  $g(x, y) = c$ , then  $(a, b)$  either satisfies the equations

$$\text{grad } f(a, b) = \lambda \text{grad } g(a, b) \text{ and } g(a, b) = c \text{ for some } \lambda \in \mathbb{R},$$

or  $\text{grad } g(a, b) = 0$ . The number  $\lambda$  is called the Lagrange multiplier.

## **Licensing**

### **Creative Commons Attribution 4.0 International**

=====  
Creative Commons Corporation (“Creative Commons”) is not a law firm and does not provide legal services or legal advice. Distribution of Creative Commons public licenses does not create a lawyer-client or other relationship. Creative Commons makes its licenses and related information available on an “as-is” basis. Creative Commons gives no warranties regarding its licenses, any material licensed under their terms and conditions, or any related information. Creative Commons disclaims all liability for damages resulting from their use to the fullest extent possible.

#### Using Creative Commons Public Licenses

Creative Commons public licenses provide a standard set of terms and conditions that creators and other rights holders may use to share original works of authorship and other material subject to copyright and certain other rights specified in the public license below. The following considerations are for informational purposes only, are not exhaustive, and do not form part of our licenses.

Considerations for licensors: Our public licenses are intended for use by those authorized to give the public permission to use material in ways otherwise restricted by copyright and certain other rights. Our licenses are irrevocable. Licensors should read and understand the terms and conditions of the license they choose before applying it. Licensors should also secure all rights necessary before applying our licenses so that the public can reuse the material as expected. Licensors should clearly mark any material not subject to the license. This includes other CC-licensed material, or material used under an exception or limitation to copyright. More considerations for licensors: [wiki.creativecommons.org/Considerations\\_for\\_licensors](http://wiki.creativecommons.org/Considerations_for_licensors)

Considerations for the public: By using one of our public licenses, a licensor grants the public permission to use the licensed material under specified terms and conditions. If the licensor’s permission is not necessary for any reason—for example, because of any applicable exception or limitation to copyright—then that use is not regulated by the license. Our licenses grant only permissions under copyright and certain other rights that a licensor has authority to grant. Use of the licensed material may still be restricted for other reasons, including because others have copyright or other rights in the material. A licensor may make special requests,

such as asking that all changes be marked or described. Although not required by our licenses, you are encouraged to respect those requests where reasonable. More considerations for the public:

[wiki.creativecommons.org/Considerations\\_for\\_licensees](http://wiki.creativecommons.org/Considerations_for_licensees)

=====  
Creative Commons Attribution 4.0 International Public License

By exercising the Licensed Rights (defined below), You accept and agree to be bound by the terms and conditions of this Creative Commons Attribution 4.0 International Public License (“Public License”). To the extent this Public License may be interpreted as a contract, You are granted the Licensed Rights in consideration of Your acceptance of these terms and conditions, and the Licensor grants You such rights in consideration of benefits the Licensor receives from making the Licensed Material available under these terms and conditions.

Section 1 – Definitions.

- a. Adapted Material means material subject to Copyright and Similar Rights that is derived from or based upon the Licensed Material and in which the Licensed Material is translated, altered, arranged, transformed, or otherwise modified in a manner requiring permission under the Copyright and Similar Rights held by the Licensor. For purposes of this Public License, where the Licensed Material is a musical work, performance, or sound recording, Adapted Material is always produced where the Licensed Material is synched in timed relation with a moving image.
- b. Adapter’s License means the license You apply to Your Copyright and Similar Rights in Your contributions to Adapted Material in accordance with the terms and conditions of this Public License.
- c. Copyright and Similar Rights means copyright and/or similar rights closely related to copyright including, without limitation, performance, broadcast, sound recording, and Sui Generis Database Rights, without regard to how the rights are labeled or categorized. For purposes of this Public License, the rights specified in Section 2(b)(1)-(2) are not Copyright and Similar Rights.
- d. Effective Technological Measures means those measures that, in the absence of proper authority, may not be circumvented under laws fulfilling obligations under Article 11 of the WIPO Copyright Treaty adopted on December 20, 1996, and/or similar international agreements.
- e. Exceptions and Limitations means fair use, fair dealing, and/or any other exception or limitation to Copyright and Similar Rights that applies to Your use of the Licensed Material.
- f. Licensed Material means the artistic or literary work, database, or other material to which the Licensor applied this Public License.

- g. Licensed Rights means the rights granted to You subject to the terms and conditions of this Public License, which are limited to all Copyright and Similar Rights that apply to Your use of the Licensed Material and that the Licensor has authority to license.
- h. Licensor means the individual(s) or entity(ies) granting rights under this Public License.
- i. Share means to provide material to the public by any means or process that requires permission under the Licensed Rights, such as reproduction, public display, public performance, distribution, dissemination, communication, or importation, and to make material available to the public including in ways that members of the public may access the material from a place and at a time individually chosen by them.
- j. Sui Generis Database Rights means rights other than copyright resulting from Directive 96/9/EC of the European Parliament and of the Council of 11 March 1996 on the legal protection of databases, as amended and/or succeeded, as well as other essentially equivalent rights anywhere in the world.
- k. You means the individual or entity exercising the Licensed Rights under this Public License. Your has a corresponding meaning.

## Section 2 – Scope.

- a. License grant.
  - 1. Subject to the terms and conditions of this Public License, the Licensor hereby grants You a worldwide, royalty-free, non-sublicensable, non-exclusive, irrevocable license to exercise the Licensed Rights in the Licensed Material to:
    - a. reproduce and Share the Licensed Material, in whole or in part; and
    - b. produce, reproduce, and Share Adapted Material.
  - 2. Exceptions and Limitations. For the avoidance of doubt, where Exceptions and Limitations apply to Your use, this Public License does not apply, and You do not need to comply with its terms and conditions.
  - 3. Term. The term of this Public License is specified in Section 6(a).
  - 4. Media and formats; technical modifications allowed. The Licensor authorizes You to exercise the Licensed Rights in all media and formats whether now known or hereafter created, and to make technical modifications necessary to do so. The Licensor waives and/or agrees not to assert any right or authority to forbid You from making technical modifications necessary to exercise the Licensed Rights, including technical modifications necessary to circumvent Effective Technological Measures. For purposes of this Public License, simply making modifications authorized by this Section 2(a)
    - (4) never produces Adapted Material.

5. Downstream recipients.
  - a. Offer from the Licensor -- Licensed Material. Every recipient of the Licensed Material automatically receives an offer from the Licensor to exercise the Licensed Rights under the terms and conditions of this Public License.
  - b. No downstream restrictions. You may not offer or impose any additional or different terms or conditions on, or apply any Effective Technological Measures to, the Licensed Material if doing so restricts exercise of the Licensed Rights by any recipient of the Licensed Material.
6. No endorsement. Nothing in this Public License constitutes or may be construed as permission to assert or imply that You are, or that Your use of the Licensed Material is, connected with, or sponsored, endorsed, or granted official status by, the Licensor or others designated to receive attribution as provided in Section 3(a)(1)(A)(i).
  - b. Other rights.
    1. Moral rights, such as the right of integrity, are not licensed under this Public License, nor are publicity, privacy, and/or other similar personality rights; however, to the extent possible, the Licensor waives and/or agrees not to assert any such rights held by the Licensor to the limited extent necessary to allow You to exercise the Licensed Rights, but not otherwise.
    2. Patent and trademark rights are not licensed under this Public License.
    3. To the extent possible, the Licensor waives any right to collect royalties from You for the exercise of the Licensed Rights, whether directly or through a collecting society under any voluntary or waivable statutory or compulsory licensing scheme. In all other cases the Licensor expressly reserves any right to collect such royalties.

### Section 3 – License Conditions.

Your exercise of the Licensed Rights is expressly made subject to the following conditions.

- a. Attribution.
  1. If You Share the Licensed Material (including in modified form), You must:
    - a. retain the following if it is supplied by the Licensor with the Licensed Material:
      - i. identification of the creator(s) of the Licensed Material and any others designated to receive



attribution, in any reasonable manner requested by the Licensor (including by pseudonym if designated);

- ii. a copyright notice;
  - iii. a notice that refers to this Public License;
  - iv. a notice that refers to the disclaimer of warranties;
  - v. a URI or hyperlink to the Licensed Material to the extent reasonably practicable;
- b. indicate if You modified the Licensed Material and retain an indication of any previous modifications; and
  - c. indicate the Licensed Material is licensed under this Public License, and include the text of, or the URI or hyperlink to, this Public License.
2. You may satisfy the conditions in Section 3(a)(1) in any reasonable manner based on the medium, means, and context in which You Share the Licensed Material. For example, it may be reasonable to satisfy the conditions by providing a URI or hyperlink to a resource that includes the required information.
  3. If requested by the Licensor, You must remove any of the information required by Section 3(a)(1)(A) to the extent reasonably practicable.
  4. If You Share Adapted Material You produce, the Adapter's License You apply must not prevent recipients of the Adapted Material from complying with this Public License.

#### Section 4 – Sui Generis Database Rights.

Where the Licensed Rights include Sui Generis Database Rights that apply to Your use of the Licensed Material:

- a. for the avoidance of doubt, Section 2(a)(1) grants You the right to extract, reuse, reproduce, and Share all or a substantial portion of the contents of the database;
- b. if You include all or a substantial portion of the database contents in a database in which You have Sui Generis Database Rights, then the database in which You have Sui Generis Database Rights (but not its individual contents) is Adapted Material; and
- c. You must comply with the conditions in Section 3(a) if You Share all or a substantial portion of the contents of the database.

For the avoidance of doubt, this Section 4 supplements and does not replace Your obligations under this Public License where the Licensed Rights include other Copyright and Similar Rights.

Section 5 – Disclaimer of Warranties and Limitation of Liability.

- a. UNLESS OTHERWISE SEPARATELY UNDERTAKEN BY THE LICENSOR, TO THE EXTENT POSSIBLE, THE LICENSOR OFFERS THE LICENSED MATERIAL AS-IS AND AS-AVAILABLE, AND MAKES NO REPRESENTATIONS OR WARRANTIES OF ANY KIND CONCERNING THE LICENSED MATERIAL, WHETHER EXPRESS, IMPLIED, STATUTORY, OR OTHER. THIS INCLUDES, WITHOUT LIMITATION, WARRANTIES OF TITLE, MERCHANTABILITY, FITNESS FOR A PARTICULAR PURPOSE, NON-INFRINGEMENT, ABSENCE OF LATENT OR OTHER DEFECTS, ACCURACY, OR THE PRESENCE OR ABSENCE OF ERRORS, WHETHER OR NOT KNOWN OR DISCOVERABLE. WHERE DISCLAIMERS OF WARRANTIES ARE NOT ALLOWED IN FULL OR IN PART, THIS DISCLAIMER MAY NOT APPLY TO YOU.
- b. TO THE EXTENT POSSIBLE, IN NO EVENT WILL THE LICENSOR BE LIABLE TO YOU ON ANY LEGAL THEORY (INCLUDING, WITHOUT LIMITATION, NEGLIGENCE) OR OTHERWISE FOR ANY DIRECT, SPECIAL, INDIRECT, INCIDENTAL, CONSEQUENTIAL, PUNITIVE, EXEMPLARY, OR OTHER LOSSES, COSTS, EXPENSES, OR DAMAGES ARISING OUT OF THIS PUBLIC LICENSE OR USE OF THE LICENSED MATERIAL, EVEN IF THE LICENSOR HAS BEEN ADVISED OF THE POSSIBILITY OF SUCH LOSSES, COSTS, EXPENSES, OR DAMAGES. WHERE A LIMITATION OF LIABILITY IS NOT ALLOWED IN FULL OR IN PART, THIS LIMITATION MAY NOT APPLY TO YOU.
- c. The disclaimer of warranties and limitation of liability provided above shall be interpreted in a manner that, to the extent possible, most closely approximates an absolute disclaimer and waiver of all liability.

Section 6 – Term and Termination.

- a. This Public License applies for the term of the Copyright and Similar Rights licensed here. However, if You fail to comply with this Public License, then Your rights under this Public License terminate automatically.
- b. Where Your right to use the Licensed Material has terminated under Section 6(a), it reinstates:
  1. automatically as of the date the violation is cured, provided it is cured within 30 days of Your discovery of the violation; or
  2. upon express reinstatement by the Licensor.

For the avoidance of doubt, this Section 6(b) does not affect any right the Licensor may have to seek remedies for Your violations of this Public License.

- c. For the avoidance of doubt, the Licensor may also offer the Licensed Material under separate terms or conditions or stop distributing the Licensed Material at any time; however, doing so will not terminate this Public License.
- d. Sections 1, 5, 6, 7, and 8 survive termination of this Public License.

Section 7 – Other Terms and Conditions.

- a. The Licensor shall not be bound by any additional or different terms or conditions communicated by You unless expressly agreed.
- b. Any arrangements, understandings, or agreements regarding the Licensed Material not stated herein are separate from and independent of the terms and conditions of this Public License.

Section 8 – Interpretation.

- a. For the avoidance of doubt, this Public License does not, and shall not be interpreted to, reduce, limit, restrict, or impose conditions on any use of the Licensed Material that could lawfully be made without permission under this Public License.
- b. To the extent possible, if any provision of this Public License is deemed unenforceable, it shall be automatically reformed to the minimum extent necessary to make it enforceable. If the provision cannot be reformed, it shall be severed from this Public License without affecting the enforceability of the remaining terms and conditions.
- c. No term or condition of this Public License will be waived and no failure to comply consented to unless expressly agreed to by the Licensor.
- d. Nothing in this Public License constitutes or may be interpreted as a limitation upon, or waiver of, any privileges and immunities that apply to the Licensor or You, including from the legal processes of any jurisdiction or authority.

=====

Creative Commons is not a party to its public licenses. Notwithstanding, Creative Commons may elect to apply one of its public licenses to material it publishes and in those instances will be considered the “Licensor.” Except for the limited purpose of indicating that material is shared under a Creative Commons public license or as otherwise permitted by the Creative Commons policies published at [creativecommons.org/policies](https://creativecommons.org/policies), Creative Commons does not authorize the use of the trademark “Creative Commons” or any other trademark or logo of Creative Commons without its prior written consent including, without limitation, in connection with any unauthorized modifications to any of its public licenses or any other arrangements, understandings, or agreements concerning use of licensed material. For the avoidance of doubt, this paragraph does not form part of the public licenses.

Creative Commons may be contacted at <https://creativecommons.org>.