Short Notes in Mathematics

Fundamental concepts every student of Mathematics should know

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Contents

1 Foundations

Set theory

1.1 Naive Definition of Sets (Georg Cantor 1882) A set is a gathering together into a whole of definite, distinct objects of our perception or of our thought which are called elements of the set.

1.2 In modern Mathematics, set theory is developed axiomatically. Axiomatic set theory goes back to Zermelo-Fraenkel and forms the basis of modern mathematics, together with formal logic. We follow the axiomatic approach here too, even though we will not introduce all of the set theory axioms of Zermelo-Fraenkel and will not go into the subtleties of set theory.

1.3 Axiomatic Definition of Sets (cf. Zermelo-Fraenkel 1908, 1973) Sets are denoted by letters of the roman and greek alphabets : $a, b, c, \ldots, x, y, z$, $A, B, C, \ldots, X, Y, Z$, $\alpha, \beta, \gamma, \delta, \ldots, \chi, \psi, \omega$, and $A, B, \Gamma, \Delta, \ldots, X, \Psi, \Omega$. The fundamental relational symbols of set theory are the *equality sign* $=$ and the *element sign* $\in$. Further symbols of set theory are the numerical constants $0, 1, 2, \ldots, 9$ and the symbol for the empty set $\varnothing$.

(S1) (Extensionality Axiom)
Two sets M and N are equal if and only if they have the same elements.
Formally: $\forall M\, \forall N\, \big((M = N) \Leftrightarrow (\forall x\, (x \in M) \Leftrightarrow (x \in N))\big)$.

(Definition of the subset relation)
A set M is called a *subset* of a set N, in signs $M \subset N$, if every element of M is an element of N.
Formally: $(M \subset N) \Leftrightarrow (\forall x\, (x \in M) \Rightarrow (x \in N))$.

The extensionality axiom can now be expressed equivalently as follows:
$$\forall M\, \forall N\, \big((M = N) \Leftrightarrow ((M \subset N)\, \&\, (N \subset M))\big) \ .$$

(S2) (Axiom of Empty Set)
There exists a set which has no elements.
Formally: $\exists E\, \forall x\, \neg(x \in E)$.

The set having no elements is uniquely determined by the extensionality axiom. It is denoted by $\varnothing$.

(S3) (Axiom of Pairings)
For all sets x and y there exists a set containing exactly x and y.
Formally: $\forall x\, \forall y\, \exists M\, \forall z\big(z \in M \Leftrightarrow (z = x \vee z = y)\big)$.

The set containing x and y as elements (and no others) is denoted $\{x, y\}$. If $x = y$, one writes $\{x\}$ for this set.

(S4) (Separation Scheme)
Let M be a set and $P(x)$ a formula (or in other words a property). Then there exists a set N whose elements consist of all $x \in M$ such that $P(x)$ holds true.
In signs: $N = \{x \in M \mid P(x)\}$.

Using the separation scheme one can define the intersection of two sets M and N as $M \cap N = \{x \in M \mid x \in N\}$ and the complement $M \backslash N$ as the set $\{x \in M \mid x \notin N\}$.

(S5) (Axiom of the Union)
Given two sets M and N there exists a set consisting of all elements of M and N (and no others).
Formally: $\forall M \, \forall N \, \exists U \, \forall x \, \big((x \in U) \Leftrightarrow (x \in M \vee x \in N)\big)$.

The set U in this formula is uniquely defined by the extensionality axiom and is called the *union of* M *and* N. It is denoted $M \cup N$.

More generally, if M is a set, then there exists a set denoted by $\bigcup M$ consisting of all elements of elements of M.
Formally: $\forall M \, \exists U \, \forall x \, \big((x \in U) \Leftrightarrow (\exists X \, (X \in M \,\&\, x \in X))\big)$.

The set U in this formula then is uniquely determined and abbreviated $\bigcup M$.

(S6) (Power Set Axiom)
For each set M there exists a set $\mathcal{P}(M)$ containing all subsets of M (and no others).
Formally: $\forall M \, \exists P \, \forall x \, \big((x \subset M) \Leftrightarrow (x \in P)\big)$.

The set P in this formula is uniquely defined by the extensionality axiom and is called the *power set of* M. It is denoted $\mathcal{P}(M)$.

(S7) (Axiom of Infinity) There exists a set which contains $\varnothing$ and with each element x also the union $x \cup \{x\}$.
Formally: $\exists M \, \big(\varnothing \in M \,\&\, \forall x (x \in M \Rightarrow x \cup \{x\} \in M)\big)$.

(S8) (Axiom of Choice) For any set M of nonempty sets, there exists a choice function f defined on M that is a map $M \to \bigcup M$ such that $f(x) \in x$ for all $x \in M$.
Formally: $\forall M \big(\varnothing \notin M \Rightarrow \exists f : M \to \bigcup M, \, \forall x \in X \, (f(x) \in x)\big)$.

1.4 Remark In the formulation of the Axiom of Choice we used the notion of a function introduced below in Definition 1.13.

1.5 Proposition *Let L, M, N be sets. Then the following statements hold true.*

(a) (commutativity)
$M \cup N = N \cup M$ *and* $M \cap N = N \cap M$.

(b) (associativity)
$M \cup (N \cup L) = (N \cup M) \cup L$ *and* $M \cap (N \cap L) = (N \cap M) \cap L$.

(c) (distributivity)
$M \cup (N \cap L) = (M \cup N) \cap (M \cup L)$ *and* $M \cap (N \cup L) = (M \cap N) \cup (M \cap L)$.

(d) $M \cap N \subset M$ *and* $M \subset M \cup N$.

(e) *If for a set X the relations $X \subset M$ and $X \subset N$ hold true, then $X \subset M \cap N$. If for a set Y the relations $M \subset Y$ and $N \subset Y$ are satisfied, then $M \cup N \subset Y$.*

(f) $M \backslash \varnothing = M$ *and* $M \backslash M = \varnothing$.

(g) $M \backslash (M \cap N) = M \backslash N$ *and* $M \backslash (M \backslash N) = M \cap N$.

(h) (de Morgan's laws)
$M\backslash(N \cup L) = (M\backslash N) \cap (M\backslash L)$ *and* $M\backslash(N \cap L) = (M\backslash N) \cup (M\backslash L)$.

Cartesian products

1.6 Definition Let X and Y be sets. For all $x \in X$ and $y \in Y$ the *pair* (x, y) is defined as the set $\big\{\{x\}, \{x, y\}\big\}$. The *cartesian product* $X \times Y$ is defined as

$$\big\{z \in \mathcal{P}(\mathcal{P}(X \cup Y)) \mid \exists x \in X\, \exists y \in Y :\ z = \{\{x\}, \{x, y\}\}\big\}.$$

1.7 Proposition *For sets X, Y and elements $x, x' \in X$ and $y, y' \in Y$ the pairs (x, y) and (x', y') are equal if and only if $x = x'$ and $y = y'$.*

1.8 Proposition *Let L, M, N be sets. Then the following distributivity laws hold true:*

(a) $L \times (M \cup N) = (L \times M) \cup (L \times N)$ *and* $(M \cup N) \times L = (M \times L) \cup (N \times L)$.

(b) $L \times (M \cap N) = (L \times M) \cap (L \times N)$ *and* $(M \cap N) \times L = (M \times L) \cap (N \times L)$.

(c) $L \times (M\backslash N) = (L \times M)\backslash(L \times N)$ *and* $(M\backslash N) \times L = (M \times L)\backslash(N \times L)$.

Relations

1.9 Definition A triple $R = (X, Y, \Gamma)$ with X and Y being sets and Γ a subset of the cartesian product $X \times Y$ is called a *relation from X to Y*. If $Y = X$, a relation (X, X, Γ) is called a *relation on X*. The set Γ is called the graph of the relation.

If $(x, y) \in \Gamma$, on says that x and y are in relation, and denotes that by $x\, R\, y$.

1.10 Remark Usually, a relation is often denoted by a symbol like for example $\sim$, $\leqslant$ or $\equiv$. The statement that x and y are in relation is then symbolically written $x \sim y$, $x \leqslant y$, $x \equiv y$, respectively.

1.11 Definition A relation $\sim$ on a set X is called an *equivalence relation* if it has the following properties:

(E1) Reflexivity
For all $x \in X$ the relation $x \sim x$ holds true.

(E2) Symmetry
For all $x, y \in X$, if $x \sim y$ holds true, then $y \sim x$ is true as well.

(E3) Transitivity
For all $x, y, z \in X$ the relations $x \leqslant y$ and $y \leqslant x$ entail $x \leqslant z$.

1.12 Definition A set X together with a relation $\leqslant$ on it is called an *ordered set*, a *partially ordered set* or a *poset* if the following axioms are satisfied:

(O1) Reflexivity
For all $x \in X$ the relation $x \leqslant x$ holds true.

(O2) Antisymmetry
If $x \leqslant y$ and $y \leqslant x$ for some $x, y \in X$, then $x = y$.

(O3) Transitivity
For all $x, y, z \in X$ the relations $x \leqslant y$ and $y \leqslant x$ entail $x \leqslant z$.

The relation $\leqslant$ is then called an *order relation* or an *order* on X.

If in addition Axiom (O4) below holds true, $(X, \leqslant)$ is called a *totally ordered set* and $\leqslant$ a *total order* on X.

(O4) Totality
For all $x, y \in X$ the relation $x \leqslant y$ or the relation $y \leqslant x$ holds true.

A total order relation $\leqslant$ on X satisfies the following law of trichotomy, where $x < y$ stands for $x \leqslant y$ and $x \neq y$:

Law of Trichotomy
For all $x, y \in X$ exactly one of the statements $x < y$, $x = y$ or $y < x$ holds true.

Functions

1.13 Definition By a *function* f one understands a triple (X, Y, Γ) consisting of

(a) a set X, called the *domain* of the function,

(b) a set Y, called the *range* or *target* of the function,

(c) a set Γ of pairs (x, y) of points $x \in X$ and $y \in Y$, called the *graph* of the function, such that for each $x \in X$ there is a unique $f(x) \in Y$ with $\big(x, f(x)\big) \in \Gamma$.

A function f with domain X, range Y and graph $\Gamma = \{(x, y) \in X \times Y \mid y = f(x)\}$ will be denoted

$$f : X \to Y,\ x \mapsto f(x)\ .$$

1.14 Example The following are examples of functions:

(a) the identity function on a set X, $\mathrm{id}_X : X \to X$, $x \mapsto x$,

(b) polynomial functions which are functions of the form $p : \mathbb{R} \to \mathbb{R}$, $x \mapsto \sum_{k=0}^{n} a_k x^k$, where the a_k, $k = 0, \ldots, n$ are real numbers called the coefficients of the polynomial function,

(c) the absolute value function $|\cdot| : \mathbb{R} \to \mathbb{R}$, $x \mapsto |x| = \sqrt{x^2}$,

the euclidean norm $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto \sqrt{x^2 + y^2}$ on $\mathbb{R}^2$ and more generally the euclidean norm $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$, $(x_1, \ldots, x_n) \mapsto \sqrt{\sum_{i=10}^{n} x_i^2}$ on $\mathbb{R}^n$.

Further examples of functions defined on (subsets of) $\mathbb{R}$ are the exponential funtion, the logarithm the trigonometric functions, and so on. Precise definitions of these will be introduced later.

1.15 Definition Functions of the form $f : \mathbb{R}^m \to \mathbb{R}^n$ such that

(L1) $f(0) = 0$ and

(L2) $f(v + w) = f(v) + f(w)$ for all $v, w \in \mathbb{R}^m$

are called *linear*. A function $g : \mathbb{R}^m \to \mathbb{R}^n$ is called *affine* if there exists an $a \in \mathbb{R}^n$ and a linear function $f : \mathbb{R}^m \to \mathbb{R}^n$ such that $g(v) = f(v) + a$ for all $v \in \mathbb{R}^m$.

1.16 Definition A function $f : X \to Y$ is called

- *injective* or *one-to-one* if for all $x_1, x_2 \in X$ the equality $f(x_1) = f(x_2)$ implies $x_1 = x_2$,
- *surjective* or *onto* if for all $y \in Y$ there exists an $x \in X$ such that $f(x) = y$, and
- *bijective* if it is injective and bijective.

1.17 Definition Let $f : X \to Y$ and $g : Y \to Z$ be two functions. The *composition* $g \circ f : X \to Z$ then is defined as the function with domain X, range Z and graph $\Gamma = \{(x, z) \in X \times Z \mid z = g(f(x))\}$. In other words, $g \circ f$ maps an element $x \in X$ to $g(f(x)) \in Z$.

1.18 Definition A function $f : X \to Y$ is called *invertible* if there exist a function $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

1.19 Theorem *A function $f : X \to Y$ is invertible if and only if it is bijective.*

2 Number Systems

Natural numbers

2.1 Definition (Peano) A triple $(\mathbb{N}, 0, s)$ consisting of a set $\mathbb{N}$, an element $0 \in \mathbb{N}$ called *zero element* and a map $s : \mathbb{N} \to \mathbb{N}$ called *successor map* is called a *system of natural numbers* if the following axioms hold true:

(P1) 0 is not in the image of s.

(P2) s is injective.

(P3) (Induction Axiom) Every inductive subset of $\mathbb{N}$ coincides with $\mathbb{N}$, where by an *inductive subset of* $\mathbb{N}$ one understands a set $I \subset \mathbb{N}$ having the following properties:

- (I1) 0 is an element of I.
- (I2) If $n \in I$, then $s(n) \in I$.

By Axiom (P1), 0 is not in the image of the successor map. But all other elements of the Peano structure are.

2.2 Proposition *Let* $(\mathbb{N}, 0, s)$ *be a system of natural numbers. Then the image of s coincides with the set* $\mathbb{N}_{\neq 0} := \{n \in \mathbb{N} \mid n \neq 0\}$ *of all non-zero elements, in signs* $s(\mathbb{N}) = \mathbb{N}_{\neq 0}$.

2.3 Theorem *The set* $\mathbb{N}$ *of natural numbers together with addition* $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, *multiplication* $\cdot : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *and the elements* 0 *and* $1 := s(0)$ *satisfies the following axioms:*

(A1) (associativity)
$l + (m + n) = (l + m) + n$ *for all* $l, m, n \in \mathbb{N}$.

(A2) (commutativity)
$m + n = n + m$ *for all* $m, n \in \mathbb{N}$.

(A3) (neutrality of 0)
$m + 0 = 0 + m = m$ *for all* $m \in \mathbb{N}$.

(M1) (associativity)
$l \cdot (m \cdot n) = (l \cdot m) \cdot n$ *for all* $l, m, n \in \mathbb{N}$.

(M2) (commutativity)
$m \cdot n = n \cdot m$ *for all* $m, n \in \mathbb{N}$.

(M3) (neutrality of 1)
$m \cdot 1 = 1 \cdot m = m$ *for all* $m \in \mathbb{N}$.

(D) (distributivity)
$l \cdot (m + n) = (l \cdot m) + (l \cdot n)$ *and*
$(m + n) \cdot l = (m \cdot l) + (n \cdot l)$ *for all* $l, m, n \in \mathbb{N}$.

In other words, $\mathbb{N}$ *together with* $+$ *and* $\cdot$ *and the elements* $0, 1$ *is a semiring.*

Real numbers

2.4 Definition By a *field of real numbers* one understands a set $\mathbb{R}$ together together with operations $+ : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ and $\cdot : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ called *addition* and *multiplication*, two distinct elements 0 and 1 and an order relation $\leqslant$ such that the following axioms are satisfied:

(A1) (associativity)
$x + (y + z) = (x + y) + z$ for all $x, y, z \in \mathbb{R}$.

(A2) (commutativity)
$x + y = y + x$ for all $x, y \in \mathbb{R}$. .

(A3) (neutrality of 0)
$x + 0 = 0 + x = x$ for all $x \in \mathbb{R}$.

(A4) (additive inverses)
For every $x \in \mathbb{R}$ there exists $y \in \mathbb{R}$, called *negative* of x such that $x + y = y + x = 0$. The negative of x is usually denoted $-x$.

(M1) (associativity)
$x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in \mathbb{R}$.

(M2) (commutativity)
$x \cdot y = y \cdot x$ for all $x, y \in \mathbb{R}$.

(M3) (neutrality of 1)
$x \cdot 1 = 1 \cdot x = x$ for all $x \in \mathbb{R}$.

(M4) (multiplicative inverses of nonzero elements)
For every $x \in \mathbb{R}^* := \mathbb{R}\backslash\{0\}$ there exists $y \in \mathbb{R}$, called *inverse* of x such that $x \cdot y = y \cdot x = 1$. The inverse of $x \neq 0$ is usually denoted x^{-1}.

(D) (distributivity)
$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ and
$(x + y) \cdot z = (x \cdot z) + (y \cdot z)$ for all $x, y, z \in \mathbb{R}$.

(O5) (monotony of addition)
For all $x, y, z \in \mathbb{R}$ the relation $x \leqslant y$ implies $x + z \leqslant y + z$.

(O6) (monotony of multiplication)
For all $x, y, z \in \mathbb{R}$ with $z \geqslant 0$ the relation $x \leqslant y$ implies $x \cdot z \leqslant y \cdot z$.

(C) (completeness)
Each non-empty subset $X \subset \mathbb{R}$ bounded above has a least upper bound.

In other words, $\mathbb{R}$ together with $+$ and $\cdot$, the elements $0, 1$ and the order relation $\leqslant$ is a complete ordered field.

2.5 Theorem *There exists (up to isomorphism) only one field of real numbers $\mathbb{R}$.*

3 Analysis

3.1 Limits and Continuity

3.2 Definition Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. The sequence is said to *converge* to a real number $x \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon \text{ for all natural } n \geqslant N \ .$$

One then calls $(x_n)_{n\in\mathbb{N}}$ a *convergent* sequence and x its *limit.*

3.3 Proposition *The limit of a convergent sequence is uniquely determined.*

3.4 Definition The function $f : \mathbb{R}^2 \to \mathbb{R}$ has the *limit* L at the point $(a, b) \in \mathbb{R}^2$, written

$$\lim_{(x,y)\to(a,b)} f(x, y) = L,$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \text{ for all } (x, y) \neq (a, b) \text{ with } d\big((x, y), (a, b)\big) < \delta.$$

3.5 Remark Intuitively, $\lim_{(x,y)\to(a,b)} f(x, y) = L$ means that $f(x, y)$ is as close to L as we wish whenever the distance of the point (x, y) to (a, b) is sufficiently small.

3.6 Definition The function $f : \mathbb{R}^2 \to \mathbb{R}$ is called *continuous* at the point $(a, b) \in \mathbb{R}^2$, if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

The function f is said to be *continuous* on a region $R \subset \mathbb{R}^2$, if it is continuous at every point $(a, b) \in R$.

3.7 Differentiability

3.8 Definition A function $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto f(x, y)$ is called *partially differentiable in the point* $(a, b) \in \mathbb{R}^2$ *with respect to the variable* x (resp. y), if the limit

$$\frac{\partial f}{\partial x}(a, b) := \lim_{h\to 0} \frac{f(a + h, b) - f(a, b)}{h}$$
$$\Big(\text{ resp. } \frac{\partial f}{\partial y}(a, b) := \lim_{h\to 0} \frac{f(a, b + h) - f(a, b)}{h} \ \Big)$$

exists. One then calls $\frac{\partial f}{\partial x}(a, b)$ and $\frac{\partial f}{\partial y}(a, b)$ the *partial derivatives* of f at (a, b). If f is partially differentiable in every point of $\mathbb{R}^2$ with respect to the variables x and y, then one says that f is *partially differentiable.*

A function $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto f(x, y)$ is called *twice partially differentiable*, if it is partially differentiable, and if the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are partially differentiable as well.

A function $f : \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto f(x,y)$ is called *differentiable in the point* $(a,b) \in \mathbb{R}^2$, if there exists a linear function $L : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\lim_{(x,y)\to(a,b)} \frac{E(x,y)}{\sqrt{(x-a)^2+(y-b)^2}} = 0,$$

where $E : \mathbb{R}^2 \to \mathbb{R}$ is the *error function* defined by $E(x,y) := f(x,y) - f(a,b) - L(x,y)$. One then calls L the *linear approximation* of f at (a,b), and writes

$$f(x,y) \approx f(a,b) + L(x,y).$$

3.9 Definition A function $f : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto f(x)$ is called *partially differentiable in the point* $a = (a_1, \cdots, a_n) \in \mathbb{R}^n$ *with respect to the variable* x_i, if for every i, $1 \leqslant i \leqslant n$ the limit

$$\frac{\partial f}{\partial x_i}(a) := \lim_{h\to 0} \frac{f(a_1, \cdots, a_i + h, \cdots, a_n) - f(a_1, \cdots, a_n)}{h}$$

exists. One then calls $\frac{\partial f}{\partial x_i}(a)$ the (*i-th*) *partial derivative* of f at a *with respect to the variable* x_i. If f is partially differentiable in every point of $\mathbb{R}^n$ with respect to the variables $x_1, \ldots, x_n$, then one says that f is *partially differentiable.*

A function $f : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto f(x)$ is called *twice partially differentiable*, if it is partially differentiable, and if the partial derivatives $\frac{\partial f}{\partial x_i}$, $1 \leqslant i \leqslant n$ are partially differentiable as well.

A function $f : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto f(x)$ is called *differentiable in the point* $a \in \mathbb{R}^n$, if there exists a linear function $L : \mathbb{R}^n \to \mathbb{R}$ such that

$$\lim_{x\to a} \frac{E(x)}{\sqrt{(x_1-a_1)^2+\ldots+(x_n-a_n)^2}} = 0,$$

where $E : \mathbb{R}^n \to \mathbb{R}$ is the *error function* defined by $E(x) := f(x) - f(a) - L(x)$. One then calls L the *linear approximation* of f at a, and writes

$$f(x) \approx f(a) + L(x).$$

3.10 Theorem *If $f : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto f(x)$ is differentiable at $a \in \mathbb{R}^n$, then f is partially differentiable and continuous at a.*

3.11 Theorem *If $f : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto f(x)$ is twice partially differentiable, and the second partial derivatives*

$$\frac{\partial^2 f}{\partial x_i \partial x_j} := \frac{\partial}{\partial x_i}\frac{\partial f}{\partial x_j}$$

are continuous, then

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

for $1 \leqslant i, j \leqslant n$.

3.12 Definition If $f : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto f(x)$ is a partially differentiable function, and $x \in \mathbb{R}^n$, the vector

$$\operatorname{grad} f(x) := \left(\frac{\partial f}{\partial x_1}(x), \cdots, \frac{\partial f}{\partial x_n}(x) \right)$$

is called the *gradient* of f at x.

Given a function $f : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto f(x)$ which is partially differentiable (up to isolated points), a point $x \in \mathbb{R}^n$ is called a *critical point* of f, if $\operatorname{grad} f(x) = 0$, or if $\operatorname{grad} f(x)$ is not defined.

3.13 Local and Global Extrema

3.14 Definition If $f : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto f(x)$ is a function, a point $a \in \mathbb{R}^n$ is called a *local maximum* (resp. *local minimum*) of f, if $f(x) \leqslant f(a)$ (resp. $f(x) \geqslant f(a)$) for all $x \in \mathbb{R}^n$ near a.

The point $a \in \mathbb{R}^n$ is called a *global maximum* (resp. *global minimum*) of f over the region $R \subset \mathbb{R}^n$, if $f(x) \leqslant f(a)$ (resp. $f(x) \geqslant f(a)$) for all $x \in R$.

3.15 Theorem *Assume that $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto f(x, y)$ is a twice continuously partially differentiable function. Suppose that (a, b) is a point where* $\operatorname{grad} f(a, b) = 0$*. Let*

$$D = \frac{\partial^2 f}{\partial x^2}(a, b) \cdot \frac{\partial^2 f}{\partial y^2}(a, b) - \left(\frac{\partial^2 f}{\partial x \partial y}(a, b) \right)^2 .$$

- *If $D > 0$ and $\dfrac{\partial^2 f}{\partial x^2}(a, b) > 0$, then f has a local minimum in a.*
- *If $D > 0$ and $\dfrac{\partial^2 f}{\partial x^2}(a, b) < 0$, then f has a local maximum in a.*
- *If $D < 0$, then f has a saddle point in a.*
- *If $D = 0$, no conclusion can be made: f can have a local maximum, a local minimum, a saddle point, or none of these in the point (a, b).*

3.16 Definition If $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto f(x, y)$ and $g : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto g(x, y)$ are functions, and $c \in \mathbb{R}$ a number, a point $(a, b) \in \mathbb{R}^2$ is called a *local maximum* (resp. *local minimum*) of f *under the constraint* $g(x, y) = c$, if $f(x, y) \leqslant f(a, b)$ (resp. $f(x, y) \geqslant f(a, b)$) for all $(x, y) \in \mathbb{R}^2$ near (a, b) which satisfy $g(x, y) = c$.

The point $(a, b) \in \mathbb{R}^2$ is called a *global maximum* (resp. *global minimum*) of f *under the constraint* $g(x, y) = c$, if $f(x, y) \leqslant f(a, b)$ (resp. $f(x, y) \geqslant f(a, b)$) for all $(x, y) \in \mathbb{R}^2$ which satisfy $g(x, y) = c$.

3.17 Theorem *Assume that $f : \mathbb{R}^2 \to \mathbb{R}$, $x \mapsto f(x)$ is a smooth function, and $g : \mathbb{R}^2 \to \mathbb{R}$, $x \mapsto g(x)$ a smooth constraint. If f has a maximum or minimum at the point (a, b) under the constraint $g(x, y) = c$, then (a, b) either satisfies the equations*

$$\operatorname{grad} f(a, b) = \lambda \operatorname{grad} g(a, b) \text{ and } g(a, b) = c \text{ for some } \lambda \in \mathbb{R},$$

or $\operatorname{grad} g(a, b) = 0$. *The number λ is called the* Lagrange multiplier.

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