FROM WEYL QUANTIZATION TO MODERN ALGEBRAIC INDEX THEORY

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1. Introduction

One of the most influencial contributions of Hermann Weyl to mathematical physics has been his paper *Gruppentheorie und Quantenmechanik* [We27] from 1927 and its extended version, the book [We28] which was published a year later and carries the same title. The main topic of this part of Hermann Weyl's work is the mathematics of quantum mechanics. After the fundamental papers by Heisenberg and Schrödinger on the foundations of quantum mechanics had appeared in the twenties of the last century this was the central question studied in mathematical physics at that time and which to a certain degree still is present in all attempts to construct mathematically rigorous theories unifying quantum mechanics and general realtivity.

In his article *Gruppentheorie und Quantenmechanik*, Hermann Weyl essentially introduced two novel aspects to the mathematics of quantum mechanics, namely the following:

(1) The representation theory of (compact) Lie groups on Hilbert spaces was applied to mathematically determine atomic spectra.
(2) A conceptually clear quantization method was proposed which associates quantum mechanical operators to classical observables which mathematically are represented by appropriate functions of the space and momentum variables. Nowadays, this quantization scheme is named after his inventor Weyl quantization.

In this paper I will elaborate only on the second aspect, since the representation theory of compact Lie groups has already been covered in detail in other contributions to these proceedings.

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Interestingwise, other than the group theoretical part in Weyl's article from 1927, his quantization method did not immediately find acceptance in the scientific community as the following part from a review by John von Neumann in Zentralblatt shows:

> *Sodann wird eine Zuordnungsvorschrift von Matrizen zu beliebigen klassischen Größen (d.h. Funktionen der Koordinaten und Impulse) vorgeschlagen. (Da sie indessen gewisse wesentliche Anforderungen, die an eine solche Zuordnung zu stellen sind – z.B. die Definität der Matrix für wesentlich nichtnegative Größen u.ä. – verletzt, hat sie sich, trotz ihres einfachen und eleganten Baues, nicht durchsetzen können.)*

Only much later after the invention of pseudodifferential operators [Hö] and deformation quantization [BFFLS] the virtue and power of Weyl quantization became fully clear. As we will see in Section 4 of this article one can namely show by using the modern language of pseudodifferential operators that Weyl quantization satisfies the axioms of a deformation quantization á la [BFFLS] (cf. [Pf98, NeTs96]). My impression is that H. Weyl with his vision for a mathematically sound quantization scheme was quite ahead of his time. The following quote from the book [We28] supports this impression:

> *Ich kann es nun einmal nicht lassen, in diesem Drama von Mathematik und Physik – die sich im Dunkeln befruchten, aber von Angesicht zu Angesicht so gerne einander verkennen und verleugnen – die Rolle des (wie ich genügsam erfuhr, oft unerwünschten) Boten zu spielen.*

Let me explain now from the point of view of a mathematician what one means by quantization. This can be seen most easily by the following diagram:

	classical mechanics		quantum mechanics
states	points x of a symplectic manifold M respectively propability distributions on M	$\xrightarrow{\text{quantization}}$	positive normed linear functionals μ on a noncommutative C^*-algebra A
observables	elements a of the Poisson algebra $(\mathcal{C}^\infty(M), \{\ ,\ \})$	$\xleftarrow[\text{classical limit}]{}$	(self-adjoint) elements a of the C^*-algebra A
measuring process	evaluation $(x, a) \mapsto a(x)$		evaluation $(\mu, a) \mapsto \mu(a)$

Generalizing Heisenberg's commutation relations, P. M. Dirac proposed in his influential book [Di, §. 21] that a quantization map $\mathfrak{q}$ which associates to every classical observable a an element $\mathfrak{q}(a)$ of an algebra of quantum mechanical observables should satisfy the following commutation relation:

$$\big[\mathfrak{q}(a), \mathfrak{q}(b)\big] = i\hbar \mathfrak{q}(\{a, b\}), \tag{1.1}$$

where $\hbar$ denotes Planck's constant divided by 2π, a, b are classical observables, and $\{-,-\}$ is the Poisson bracket. As Dirac noticed, the commutation relations (1.1) show that "classical mechanics may be regarded as the limiting case of quantum mechanics when $\hbar$ tends to zero" (cf. [Di, §. 21]). For physical reasons Dirac's quantization conditions are usually supplemented by the requirement that the algebra of quantum observables $\mathfrak{q}(a)$ acts irreducibly on a Hilbert space $\mathcal{H}$. This Hilbert space $\mathcal{H}$ or more precisely the corresponding projective space $\mathbb{P}\mathcal{H}$ of rays in $\mathcal{H}$ is then interpreted as the space of (pure) states of the quantum mechanical system.

In 1946 it has been observed by Groenewold [Gr] and later refined by van Hove [Ho] that for the algebra of (polynomial) observables on $\mathbb{R}^{2n}$ with its standard Poisson bracket a quantization map fulfilling Dirac's commutation relations Eq. (1.1) together with the irreducibility condition cannot exist. The theorems by Groenewold–van Hove were extended by Gotay et al. [GoGrHu, Go] to more general symplectic manifolds. By all these no go results the question arises, what conditions a reasonable quantization theory should satisfy then.

Weyl's quantization scheme motivated the right answer to that problem. As it has been pointed out by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer in [BFFLS], one should regard quantization as a formal deformation of the algebra of classical observables on a symplectic manifold in the sense of Gerstenhaber [Ge]. This means that Dirac's quantization condition is required to hold only up to higher order in $\hbar$. Weyl quantization satisfies this requirement and thus provides an important example of a deformation quantization.

The paper [BFFLS] initiated quite an amount of research on the existence and uniqueness of deformation quantizations. The most outstanding are probably the existence theorem for deformation quantizations over a symplectic manifold by deWilde–Lecomte [deWiLe], the geometric and intuitive construction of star products in the symplectic case by Fedosov [Fe94], and the result on the existence and the classification of deformation quantizations for Poisson manifolds by Kontsevich [Ko]. For a detailed overview on this see for example [DiSt].

2. Weyl's commutation relations

In his analysis of quantization Weyl started from the Heisenberg commutation relations

$$\big[P, Q\big] = -i\hbar, \tag{2.1}$$

where Q resp. P denotes the quantum mechanical space resp. momentum operator. Weyl showed that these relations cannot be realized by bounded operators on a Hilbert space. His idea was then to integrate the Heisenberg commutation relations which leads to the relations

$$V(s)\, U(t) = e^{-ist\hbar} U(t) V(s), \quad s, t \in \mathbb{R}, \tag{2.2}$$

where $V(s) = e^{isQ}$ is the unitary abelian group generated by Q, and $U(t) = e^{itP}$ the one generated by P. For the Schrödinger representation on $L^2(\mathbb{R})$ given by

$$Q\, u(x) = x\, u(x), \quad P\, u(x) = -i\hbar \frac{du}{dx}(x) \quad \text{for } u \in \mathcal{S}(\mathbb{R}) \text{ and } x \in \mathbb{R} \tag{2.3}$$

one knows that

$$\big(V(s)u\big)(x) = e^{isx}u(x) \quad \text{and} \quad \big(U(t)u\big)(x) = u(x + \hbar t). \tag{2.4}$$

One thus obtains the integrated Schrödinger representation which obviously satisfies the Weyl commutation relations (2.2). Up to unitary equivalence, the integrated Schrödinger representation is the only irreducible nontrivial representation of the Weyl commutation relations. Note that the Heisenberg commutation relations have more than just one equivalence class of irreducible representations by (necessarily unbounded) symmetric operators on a Hilbert space (see [Schm]).

Using the integrated Schrödinger representation let us define now the following projective representation of $\mathbb{R}^2$:

$$W(s,t) = e^{-\frac{i}{2}\,st}U(t)\,V(s). \tag{2.5}$$

For $a \in \mathcal{S}(\mathbb{R}^2)$, the space of Schwarz test functions on $\mathbb{R}^2$, define its Weyl quantization $\mathfrak{q}_{\mathrm{W}}(a) : \mathcal{C}^\infty_{\mathrm{cpt}}(\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R})$ by

$$\big\langle v, \mathfrak{q}_{\mathrm{W}}(a)u\big\rangle := \int_{\mathbb{R}^2} \hat{a}(s,t)\langle v, W(s,t)u\rangle\, ds\, dt, \quad u, v \in \mathcal{C}^\infty_{\mathrm{cpt}}(\mathbb{R}), \tag{2.6}$$

where $\hat{a}$ denotes the Fourier transform of a. This is the original form of Weyl quantization. Let us rewrite it in a more convenient form by applying the transformation rule and Fourier transformation:

$$\begin{aligned}
\langle v, \mathfrak{q}_{\mathrm{W}}(a)u\rangle &= \int_{\mathbb{R}^2} \hat{a} \int_{\mathbb{R}} \overline{v}(x)\big(W(s,t)u\big)(x)\, dx\, ds\, dt \\
&= \int_{\mathbb{R}^3} \hat{a}(s,t)\overline{v}(x)\, e^{-\frac{i}{2}\,st\hbar}\big(U(t)V(s)u\big)(x)\, dx\, ds\, dt \\
&= \int_{\mathbb{R}^3} \hat{a}(s,t)\overline{v}(x)\, e^{-\frac{i}{2}\,st\hbar}\, e^{is(x+t\hbar)}\, u(x + t\hbar)\, dx\, ds\, dt \\
&= \frac{1}{\hbar}\int_{\mathbb{R}^3} \hat{a}(s, \frac{t}{\hbar})\overline{v}(x)e^{is(x+t/2)}\, u(x+t)\, dx\, ds\, dt \\
&= \frac{1}{2\pi\hbar}\int_{\mathbb{R}^3} a\big(x + \frac{t}{2}, \xi\big)\, \overline{v}(x)e^{-\frac{i}{\hbar}t\xi}u(x+t)\, dx\, d\xi\, dt \\
&= \frac{1}{2\pi\hbar}\int_{\mathbb{R}^3} \overline{v}(x)e^{-\frac{i}{\hbar}t\xi}a\Big(\frac{x}{2} + \big(\frac{x+t}{2}, \xi\big)\Big)\, u(x+t)\, dx\, d\xi\, dt \\
&= \frac{1}{2\pi\hbar}\int_{\mathbb{R}^3} \overline{v}(x)e^{\frac{i}{\hbar}(x-y)\xi}a\big(\frac{x+y}{2}, \xi\big)u(y)\, dx\, dy\, d\xi \\
&= \frac{1}{2\pi\hbar}\Big\langle v, \int_{\mathbb{R}^2} e^{\frac{i}{\hbar}(\bullet - y)\xi}a\big(\frac{\bullet + y}{2}, \xi\big)\, u(y)\, dy\, d\xi\Big\rangle,
\end{aligned}$$

hence

$$\big[\mathfrak{q}_{\mathrm{W}}(a)u\big](x) = \frac{1}{2\pi\hbar}\int_{\mathbb{R}^2} e^{\frac{i}{\hbar}(x-y)\xi}a\big(\frac{x+y}{2}, \xi\big)\, u(y)\, dy\, d\xi, \tag{2.7}$$

which is the form of the Weyl quantization as it usually can be found in the literature. As one checks immediatley, $\mathfrak{q}_{\mathrm{W}}(a)$ is a densly defined (in general unbounded) linear operator on $L^2(\mathbb{R})$ which is symmetric, in case a is a real-valued function. If one interprets the right hand side of Eq. (2.7) as an oscillatory integral (see [GrSj]), then Eq. (2.7) defines even for symbols $a \in \mathrm{S}^\infty(\mathbb{R})$ (cf. Sec. 4) a quantized observable $\mathfrak{q}_{\mathrm{W}}(a)$ which by definition then is a pseudodifferential operator on $\mathbb{R}$. By a standard argument in pseudodifferential calculus one shows that

$$\mathfrak{q}_{\mathrm{W}}(a)\,\mathfrak{q}_{\mathrm{W}}(a) - \mathfrak{q}_{\mathrm{W}}(b)\,\mathfrak{q}_{\mathrm{W}}(b) = -i\hbar\mathfrak{q}_{\mathrm{W}}(\{a,b\}) + o(\hbar^2) \quad \text{for } a,b \in \mathrm{S}^\infty,$$

which means that Weyl quantization satisfies Dirac's quantization condition up to higher orders in $\hbar$ or in other words that the algebra of pseudodifferential operators is a deformation of the algebra of symbols in direction of the Poisson bracket. Let us now explain the concept of a deformation quantization in some more detail.

3. Deformation quantization

Definition 3.1 ([BFFLS])**.** By a deformation quantization of a symplectic manifold (M,ω) one understands an associative and $\mathbb{C}[[\hbar]]$-bilinear product $\star$ on the space $\mathcal{A}^\hbar := \mathcal{C}^\infty(M)[[\hbar]]$ of formal power series in the (now formal) variable $\hbar$ and with coefficients in the space $\mathcal{C}^\infty(M)$ such that the following axioms hold true:

(1) There exist bidifferential operators c_k on M such that $a\star b = \sum_{k=0}^\infty c_k(a,b)\,\hbar^k$ for all $a,b \in \mathcal{C}^\infty(M)$ and such that c_0 is the commutative pointwise product of smooth functions on M.
(2) One has $a \star 1 = 1 \star a = a$ for all $a \in \mathcal{C}^\infty(M)$.
(3) The commutation relation

$$[a,b]_\star = -i\hbar\{a,b\} + o(\hbar^2)$$

is satisfied for all $a,b \in \mathcal{C}^\infty(M)$, where $[a,b]_\star := a\star b - b\star a$.

The product $\star$ is also called a star-product on M.

Example 3.2. Consider a finite dimensional symplectic vector space (V,ω), and let

$$\{-,-\} : \mathcal{C}^\infty(V)\otimes\mathcal{C}^\infty(V) \to \mathcal{C}^\infty(V), \quad a\otimes b \mapsto \sum_{1\le i,j\le \dim V} \Pi_{ij}\frac{\partial a}{\partial x_i}\frac{\partial b}{\partial x_j}$$

be its Poisson bracket, where $(x_i)_{1\le i\le \dim V}$ denote some coordinates of V. Since the standard Poisson bivector $\Pi := \sum \Pi_{ij}\frac{\partial}{\partial x_i}\otimes\frac{\partial}{\partial x_j}$ is constant, the operator

$$\hat{\Pi} : \mathcal{C}^\infty(V)\otimes\mathcal{C}^\infty(V) \to \mathcal{C}^\infty(V)\otimes\mathcal{C}^\infty(V), \quad a\otimes b \to \sum_{1\le i,j\le 2n} \Pi_{ij}\frac{\partial a}{\partial x_i}\otimes\frac{\partial b}{\partial x_j}$$

is well-defined. Denoting by μ the pointwise product of functions, one can now put

$$\star : \mathcal{C}^\infty(V)[[\hbar]]\otimes\mathcal{C}^\infty(V)[[\hbar]] \to \mathcal{C}^\infty(V)[[\hbar]], \quad a\otimes b \mapsto \sum_{k\in\mathbb{N}}\frac{(-i\hbar)^k}{k!}\mu\big(\hat{\Pi}^k(a\otimes b)\big) \tag{3.1}$$

and thus obtains a star product on $\mathcal{C}^\infty(V)$, the so-called Weyl–Moyal-product. It is immediately checked that $(\mathcal{C}^\infty(V)[[\hbar]],\star)$ is a deformation qunatization in the above sense.

By construction, the Weyl–Moyal-product makes sense also on the space $\mathbb{W}V$ of formal power series in $\hbar$ with formal power series at the origin of V as coefficients. One calls the resulting algebra $(\mathbb{W}V,\star)$ the formal Weyl algebra of V, and obtains an epimorphism of algebras $(\mathcal{C}^\infty(V)[[\hbar]],\star) \to (\mathbb{W}V,\star)$ given by formal power series expansion at the origin in each degree of $\hbar$.

The existence of a star product on an arbitrary symplectic manifold was an open mathematical problem for almost ten years after the article [BFFLS] had appeared, and was settled by independant methods in the papers [deWiLe] and [Fe94]. Another ten years passed until the existence of star products on Poisson could be proved in [Ko].

Let us briefly sketch the main idea of the proof by Fedosov [Fe96], since his approach contains essential tools which lead to algebraic index theory. Consider a symplectic manifold (M, ω) and its tangent bundle TM. Since each of the tangent spaces T_pM, $p \in M$ is a symplectic vector space, one can form the bundle of formal Weyl algebras $\mathbb{W}M := \bigsqcup_{p\in M} \mathbb{W}T_pM$. Note that the bundle of formal Weyl agebras is well-defined because the symplectic group acts as automorphisms on the formal Weyl algebra. Now consider the bundle of forms $\Lambda^\bullet\mathbb{W}M := \mathbb{W}M \otimes \Lambda^\bullet M$. Its space of smooth sections $\Omega^\bullet\mathbb{W}(M)$ obviously is a noncommutative algebra with product denoted by $\bullet$. The fundamental observation by Fedosov was that for an appropriate flat graded derivation D with respect to the product $\bullet$ on $\Omega^\bullet\mathbb{W}(M)$ the subalgebra

$$\mathcal{W}_D(M) := \{s \in \Omega^0\mathbb{W}(M) \mid Ds = 0\}$$

of flat sections is linearly isomorphic to the space of formal power series $\mathcal{C}^\infty(M)[[\hbar]]$. Via the resulting isomorphism $\mathfrak{q} : \mathcal{C}^\infty(M)[[\hbar]] \to \mathcal{W}_D(M)$ one can then push down the product on $\mathcal{W}_D(M)$ to $\mathcal{C}^\infty(M)[[\hbar]]$ and thus obtains a star product on M. The flat connection needed for this construction has the form

$$D = \nabla + [A, -],$$

where ∇ is a symplectic connection on M, i.e. $\nabla\omega = 0$, $[-, -]$ is the commutator with respect to the product $\bullet$, and $A \in \Omega^1\mathbb{W}(M)$. The cohomology class of the curvature

$$\Omega := \nabla A + \frac{1}{2}[A, A]$$

(note that it is a formal power series in $\hbar$) classifies the star product $\star$ up to equivalence.

For the application of deformation quantization to index theory, the notion of a trace on a deformation quantization $\left(\mathcal{C}^\infty_{\text{cpt}}(M)[[\hbar]], \star\right)$ is crucial. By a that one understands a linear functional $\text{tr} : \mathcal{C}^\infty_{\text{cpt}}(M)[[\hbar]] \to \mathbb{C}[[\hbar, \hbar^{-1}]$ which vanishes on commutators. The following result provides essential information on the existence and uniqueness of such traces.

Proposition 3.3 ([NeTs95, Fe96]). *The space of traces on a deformation quantization $\left(\mathcal{C}^\infty_{\text{cpt}}(M)[[\hbar]], \star\right)$ over a connected symplectic manifold M is one-dimensional.*

4. Pseudodifferential operators

Next we will set up Weyl quantization within the language of pseudodifferential operators. Before we come to the details of this let us recall some basics of that theory.

Let $U \subset \mathbb{R}^n$ be open. By a symbol on $U \times \mathbb{R}^N$ of order $m \in \mathbb{R}$ one understands a function $a \in \mathcal{C}^\infty(U \times \mathbb{R}^N)$ such that for every compact $K \subset U$ and all multi-indices $\alpha \in \mathbb{N}^n$ and $\beta \in \mathbb{N}^N$ there exists a $C_{K,\alpha,\beta} > 0$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)\right| \le C_{K,\alpha,\beta}\,(1 + ||\xi||)^{m-|\beta|} \quad \text{for all } (x, \xi) \in K \times \mathbb{R}^N. \tag{4.1}$$

The space of such symbols is denoted by $\mathrm{S}^m(U, \mathbb{R}^N)$. Obviously, one can easily generalize the notion of a symbol of order m to smooth maps $a : E \to \mathbb{R}$ defined on a vector bundle $E \to M$ by requiring Eq. (4.1) to hold locally in bundle charts. For every manifold X we denote the space of symbols of order m on the cotangent bundle T^*X by $\mathrm{S}^m(X)$. Moreover, one puts $\mathrm{S}^\infty(X) := \bigcup_{m\in\mathbb{R}} \mathrm{S}^m(X)$ and $\mathrm{S}^{-\infty}(X) := \bigcap_{m\in\mathbb{R}} \mathrm{S}^m(X)$.

A pseudodifferential operator over $U \subset \mathbb{R}^n$ now is a linear operator $A : \mathcal{C}^\infty_{\mathrm{cpt}}(U) \to \mathcal{C}^\infty(U)$ which can be represented as an oscillatory integral (cf. [GrSj, Sec. 1])

$$Au(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_U e^{i\langle x-y,\xi\rangle} a(x, y, \xi)\, u(y)\, dy\, d\xi, \tag{4.2}$$

where $u \in \mathcal{C}^\infty_{\mathrm{cpt}}(U)$ and $a \in \mathrm{S}^m(U \times U, \mathbb{R}^n)$ for some $m \in \mathbb{R} \cup \{\pm\infty\}$. The space of thus defined pseudodifferential operators of order m will be denoted by $\Psi^m(U)$. More generally, if X is a manifold, the space $\Psi^m(X)$ of pseudodifferential operators of order m on X consists of all linear operators $A : \mathcal{C}^\infty_{\mathrm{cpt}}(X) \to \mathcal{C}^\infty(X)$ which can be written in the form

$$Au = A_0 u + \sum_{j\in J} \varphi_j \Big(A_j \big((\varphi_j u) \circ x_j^{-1}\big)\Big) \circ x_j,$$

where the x_j, $j \in J$ run through an atlas of X, $(\varphi_j)_{j\in J}$ is a locally finite smooth partition of unity subordinate to the domains of the charts φ_j, the A_j are pseudodifferential operators on $\mathbb{R}^{\dim X}$ of order m, and finally A_0 is a smoothing operator, which means that its Schwartz kernel is smooth. Let us restrict our considerations now to the space $\Psi^\infty_{\mathrm{ps}}(X)$ of properly supported pseudodifferential which means of all pseudodifferential operators A such that the projections $\mathrm{pr}_{1/2} : \operatorname{supp} K_A \to X$ of the support of the Schwartz kernel of A on the first resp. second coordinate are proper maps. Since every properly supported pseudodifferential operator maps functions with compact support to functions with compact support, $\Psi^\infty_{\mathrm{ps}}(X)$ turns out to be an algebra which, as we will see in the following, can be interpreted as a quantization of the symbol algebra on TX.

Let us provide some details. After the choice of a riemannian metric on X and fixing an ordering parameter $s \in [0, 1]$, define for every symbol $a \in \mathrm{S}^m(X)$ and $\hbar \in \mathbb{R}^*$ a quantization $\mathfrak{q}_s(a) : \mathcal{C}^\infty_{\mathrm{cpt}}(X) \to \mathcal{C}^\infty_{\mathrm{cpt}}(X)$ by

$$[\mathfrak{q}_s(a)u](x) = \frac{1}{(2\pi\hbar)^n} \int_{T^*X} \chi(x, y)\, e^{\frac{i}{\hbar}\langle \exp_y^{-1}(x),\xi\rangle} a\big(\tau_{g_s(x,y),y}\xi\big) u(y)\, dy d\xi. \tag{4.3}$$

The ingredients of this formula are given as follows. As usual, exp denotes the exponential function with respect to the riemannian metric on X, and χ is a properly supported cut-off function around the diagonal of $X \times X$ such that $\exp_y^{-1}(x)$ is defined for all $(x, y) \in \operatorname{supp} \chi$. By $g_s(x, y)$ we mean the s-midpoint between x and y, or in other words the point $\exp\big(s \cdot \exp_x^{-1}(y)\big)$. For x and y close enough we denote by $\tau_{x,y}$ the parallel transport in T^*X from T^*_yX to T^*_xX along the geodesic joining x and y. Finally, $dy\, d\xi$ stands for the Liouville volume element on the symplectic manifold T^*X. One checks immediately (cf. [Pf98, Vo]) that $\mathfrak{q}_s(a)$ is a (properly supported) pseudodifferential operator of order m. In case $s = 0$ one calls it the standard order quantization of the symbol a, if $s = \frac{1}{2}$, one obtains Weyl quantization on the riemannian manifold X. The reader is invited to check that on the cotangent bundle of $\mathbb{R}$, $\mathfrak{q}_{\frac{1}{2}}$ coincides with the Weyl quantization $\mathfrak{q}_{\mathrm{W}}$ from Eq. (2.7) (up to some negligible smoothing operator).

The quantization map $\mathfrak{q}_s$ has a pseudoinverse, namely the symbol map $\sigma_s : \Psi^\infty_{\mathrm{ps}}(X) \to \mathrm{S}^\infty(X)$ which is defined by

$$\sigma_s(A)(x,\xi) = \int_{T_xX} \chi_s(x,v)\, e^{\frac{i}{\hbar}\langle v,\xi\rangle}\, K_A\big(\exp_x(-sv), \exp_x((1-s)v)\big)\, \rho_s(x,v) d\theta_x(v), \tag{4.4}$$

where K_A is the Schwartz kernel of the operator A, the cut-off function χ_s is defined by $\chi_s(x,v) := \chi\big(\exp_x(-sv), \exp_x((1-s)v)\big)$, θ_x is the euclidean volume element of T_xX induced by the riemannian metric on X, and the metric factor ρ_s satisfies $\rho_s(x,v) = \rho\big(\exp_x(-sv), \exp_x((1-s)v)\big)$ with $\rho(x, \exp_x v)\theta_x = \big(\exp^*_x \mu\big)(v)$, and μ the riemannian volume element on X. Then one has

$$\sigma_s \circ \mathfrak{q}_s(a) - a \in \mathrm{S}^{-\infty}(X) \quad \text{and} \quad \mathfrak{q}_s \circ \sigma_s(A) - A \in \Psi^{-\infty}_{\mathrm{ps}}(X) \tag{4.5}$$

for all symbols a and pseudodifferential operators A on X, which shows that they are inverse to each other up to smoothing operators resp. symbols. Moreover, one can prove (cf. [PF98]) that

$$[\mathfrak{q}_s(a), \mathfrak{q}_s(b)] = -i\hbar\, \mathfrak{q}_s(\{a,b\}) + o(\hbar^2) \tag{4.6}$$

for all symbols a, b. This means that each $\mathfrak{q}_s$ and in particular Weyl quantization $\mathfrak{q}_{\mathrm{W}} := \mathfrak{q}_{\frac{1}{2}}$ induces a deformation quantization of the cotangent bundle TX.

Another usefull feature of the quantization $\mathfrak{q}_s$ is that it allows to compute the operator trace of $\mathfrak{q}_s(a)$ for every symbol a of order $m < \dim X$. According to [PF98, VO] this trace is given by

$$\operatorname{tr} \mathfrak{q}_s(a) = \frac{1}{(2\pi\hbar)^{\dim X}} \int_{T^*X} a\, \omega^{\dim X}, \tag{4.7}$$

where ω denotes the canonical symplectic form on the cotangent bundle T^*X.

5. The algebraic index theorem

Let us first recall some basic notions from index theory of Fredholm operators. Assume to be given two Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ and a Fredholm operator $F : \mathcal{H}_1 \to \mathcal{H}_2$, which means that F is a linear operator which has finite dimensional kernel and cokernel. Its index is then defined as the integer

$$\operatorname{ind} F := \dim\ker F - \dim\operatorname{coker} F. \tag{5.1}$$

The index has the following crucial properties:

- it is homotopy invariant, i.e.
$$\operatorname{ind} F(0) = \operatorname{ind} F(1)$$
for every continuous path $F : [0,1] \to \mathrm{Fred}(\mathcal{H}_1, \mathcal{H}_2)$ of Fredholm operators,
- it is additive with respect to composition, i.e.
$$\operatorname{ind}(F_1 \circ F_2) = \operatorname{ind} F_1 + \operatorname{ind} F_2$$
for two composable Fredholm operators F_1 and F_2, and finally
- the index is invariant under compact perturbations, i.e.
$$\operatorname{ind}(F + K) = \operatorname{ind} F$$
for every Fredholm operator and every compact operator K from $\mathcal{H}_1$ to $\mathcal{H}_2$.

Every Fredholm operator $F : \mathcal{H}_1 \to \mathcal{H}_2$ has a pseudoinverse $R : \mathcal{H}_2 \to \mathcal{H}_1$, which in other words means that $\mathrm{id}_{\mathcal{H}_1} - R \circ F$ and $\mathrm{id}_{\mathcal{H}_2} - R \circ F$ are both compact operators. One can even choose R such that these operators are of trace class. Then one can compute the index of F by the following formula (cf. [Fe96]):

$$\operatorname{ind} F = \operatorname{tr}(\mathrm{id}_{\mathcal{H}_1} - R \circ F) - \operatorname{tr}(\mathrm{id}_{\mathcal{H}_2} - R \circ F). \tag{5.2}$$

In the theory of linear partial differential equations, Fredholm operators appear abundantly. Namely, if $E \to X$ is a (metric) vector bundle over a compact (riemannian) manifold X, and $D : \Gamma^\infty(E) \to \Gamma^\infty(E)$ an elliptic differential operator (which means that its principle symbol is invertible) then it induces a Fredholm operator between appropriate Sobolev completions of $\Gamma^\infty(E)$. In particular, D then has a finite index, and this index does not depend on the particular choice of a Sobolev completion. By the celebrated index theorem of Atiyah–Singer [AtSi], the index of D can be computed by topological data as follows:

$$\operatorname{ind} D = (-1)^{\dim X} \int_X \operatorname{Ch}(\sigma_{\mathrm{p}}(D)) \operatorname{td}(T_{\mathbb{C}} X), \tag{5.3}$$

where $\sigma_{\mathrm{p}}(D)$ denotes the principal symbol of the differential operator D, Ch its Chern character and $\operatorname{td}(T_{\mathbb{C}} X)$ is the Todd class of the complexified tangent bundle.

As it has been observed by Fedosov [Fe96] and Nest–Tsygan [NeTs95], an "algebraic" version of this index theorem can be formulated and proved within the framework of deformation quantization. Recall that the the index of an elliptic operator can be computed by Eq. (5.2). If one interprets a deformation quantization as a kind of "formal pseudodifferential calculus", it makes sense to consider elliptic elements in the deformed algebra and define an algebraic index for these objects. More precisely, given a symplectic manifold M with a star product $\star$, one understands by an elliptic pair in $\mathcal{C}^\infty(M)[[\hbar]]$ a pair of projections P, Q in the matrix algebra over $(\mathcal{C}^\infty(M)[[\hbar]], \star)$ such that the difference $P - Q$ has compact support. This means in particular, that every elliptic pair (P, Q) determines an element $[P] - [Q]$ of the K-theory of the deformed algebra. The algebraic index of the K-theory class $[P] - [Q]$ of an elliptic pair is defined by

$$\operatorname{ind}_{\mathrm{a}} \big([P] - [Q]\big) := \operatorname{tr}(P - Q), \tag{5.4}$$

where tr is the (up to normalization) unique trace on the matrix algebra over $\mathcal{C}^\infty(M)[[\hbar]]$ (cf. Prop. 3.3). Note that the index is indeed well-defined on the K-theory of $\mathcal{C}^\infty(M)[[\hbar]]$.

The space of equivalence classes $[P] - [Q]$ of elliptic pairs is isomorphic to the space of equivalence classes of elliptic quadruples. These objects were introduced by Fedosov [Fe96] and are the natural generalizations of elliptic operators to star product algebras. More precisely, an elliptic quadruple is a quadruple $(D, F, \tilde{P}, \tilde{Q})$ of elements of the matrix algebra over $\mathcal{C}^\infty(M)[[\hbar]]$ such that the following holds:

(1) $\tilde{P}$ and $\tilde{Q}$ are projections.
(2) The elements $\tilde{P} - D \star R$ and $\tilde{Q} - R \star D$ have both compact support.

The element D of an elliptic quadruple hereby generalizes an elliptic pseudodifferential operator on a closed manifold, and F can be interpreted as its quasi-inverse.

By Eq. (5.2), the following definition of the algebraic index of an (equivalence class of an) elliptic quadruple appears to be reasonable:

$$\operatorname{ind}_{\mathrm{a}} \big([D, F, \tilde{P}, \tilde{Q}]\big) := \operatorname{tr}(\tilde{Q} - R \star D) - \operatorname{tr}(\tilde{P} - D \star R). \tag{5.5}$$

In fact, one can show that the thus defined algebraic index is compatible with the algebraic index on elliptic pairs under the mentioned isomorphism between equivalence classes of elliptic pairs and of elliptic quadruples.

There is another remark in order, here. For an elliptic pair (P, Q) the coefficients (P_0, Q_0) of order 0 in the expansion in powers of $\hbar$ are obviously projections in the matrix algebra over $\mathcal{C}^\infty(M)$, and the virtual bundle $[P_0] - [Q_0]$ defines a K-theory class of M. This map turns out to be an isomorphism between K-theories, which shows that K-theory is invariant under deformation (cf. [Ro], [Fe96, Sec. 6.1]).

The main result of algebraic index theory is the theorem below. By application of this algebraic index theorem to cotangent bundles of compact riemannian manifolds and the deformation quantization induced by Weyl quantization one obtains another proof of the index formula by Atiyah–Singer.

Theorem 5.1 ([NeTs95, Fe96])**.** *Let M be a symplectic manifold with a deformation quantization $\star$. The algebraic index of an elliptic pair $[P] - [Q]$ is then given by*

$$\mathrm{ind_a}\big([P] - [Q]\big) = \int_M \mathrm{Ch}\big([P_0] - [Q_0]\big)\, \exp\Big(-\frac{\Omega}{2\pi\hbar}\Big)\, \hat{A}(M), \tag{5.6}$$

where $\hat{A}(M)$ denotes the $\hat{A}$-genus of M, and Ω the characteristic class of the star product on M

The proof of the algebraic index theorem is quite involving. In the approach by Nest–Tsygan, methods from cyclic homology theory and Lie algebra cohomology are used intensively. We refer to the original literature for that.

6. The algebraic index theorem for orbifolds

The problem to generalize index theorems to spaces more general than (compact) manifolds has been an active area of mathematical research since many years. For orbifolds, a class of singular spaces which has attained much interest in geometry and mathematical physics, an algebraic index theorem can be proved.

In local charts, orbifolds are represented as quotients of manifolds by finite groups. Globally, and that is the approach we use in our setup, orbifolds can be presented as orbits of proper étale Lie groupoids G (see [Mo] for details). The concepts of a deformation quantization, of vector bundles, and of K-theory can all be generalized to orbifolds by requiring the objects (like a star product or a vector bundle) to be invariant on the representing proper étale groupoid. For example, the orbifold K-theory $K^0_{\mathrm{orb}}(X)$ of an orbifold X consists of equivalence classes of equivariant virtual bundles on the representing groupoid.

A quite usefull object associated to an orbifold X is its inertia orbifold $\tilde{X}$. Locally, $\tilde{X}$ consists of all fixed point manifolds of the locally representing orbifold charts. The inertia orbifold carries a lot of information about the singularities of the orbifold. The connected components of the inertia orbifold $\tilde{X}$ are sometimes called the sectors of the orbifold X.

The orbifold case is different to the manifold case in particular by one important aspect. The dimension of traces on a deformation quantization on a symplectic orbifold is in general not one (even if X is connected), but given by the number of sectors [NePfPoTa]. This means that one has to single out a particular trace to define the algebraic index for a deformation quantization on an orbifold. Fortunately, there exists a kind of "universal" trace on an orbifold, which captures from each sector a normalized contribution. With that universal trace the following algebraic index formula can be proved.

Theorem 6.1 ([PfPoTa]). *Let M be a symplectic orbifold presented by a proper étale Lie groupoid G carrying a G-invariant symplectic form ω. Let $\star$ be a star product on M, and let E and F be G-vector bundles which are isomorphic outside a compact subset of M. Then the following formula holds for the index of $[E]-[F] \in K^0_{\mathrm{orb}}(M)$:*

$$\mathrm{tr}_*\big([E]-[F]\big) = \int_{\tilde{M}} \frac{1}{m} \frac{\mathrm{Ch}_\theta\left(\frac{R^E}{2\pi i} - \frac{R^F}{2\pi i}\right)}{\det\left(1-\theta^{-1}\exp(-\frac{R^\perp}{2\pi i})\right)} \hat{A}\Big(\frac{R^\perp}{2\pi i}\Big) \exp\Big(-\frac{\iota^*\Omega}{2\pi i\hbar}\Big), \tag{6.1}$$

where $\mathrm{tr} : \mathcal{C}^\infty_{\mathrm{cpt}}[[\hbar]] \rtimes \mathsf{G} \to \mathbb{C}[[\hbar, \hbar^{-1}]$ is the universal trace on the convolution algebra capturing from each sector one contribution, m is a locally constant combinatorial function measuring the order of the isotropy group, and Ω is the characteristic class of the deformation quantization. The symbol θ denotes the action of the local isotropy groups, and $\mathrm{Ch}_\theta\left(\frac{R^E}{2\pi i} - \frac{R^F}{2\pi i}\right)$ is the equivariant Chern character which à la Chern-Weil is determined by equivariant curvatures R^E and R^F. Finally $R^\perp$ denotes the curvature of the normal bundle of the local embedding of $\tilde{X}$ into X.

Like in the manifold case one can construct a symbol calculus and Weyl quantization for orbifolds. Since Weyl quantization on an orbifold X defines a deformation quantization over the symplectic orbifold T^*X, one can then derive an analytic index formula from the algebraic index theorem for orbifolds. One then obtains the Kawasaki index formula for orbifolds (see [Ka] and [Fa]).

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