REGULARIZED TRACES AND K-THEORY INVARIANTS OF PARAMETRIC PSEUDODIFFERENTIAL OPERATORS

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Abstract. This expository article outlines our recent construction of invariants of relative K-theory classes of multi-parameter dependent pseudodifferential operators, which recover and generalize Melrose's divisor flow. These are 'higher' divisor flows, that are obtained by pairing relative K-theory classes with relative cyclic cocycles manufactured out of regularized traces. They take integral values and can be interpreted as 'suspended' versions of the spectral flow.

Introduction

Let M be a smooth compact Riemannian manifold without boundary, and let E be a Hermitian vector bundle over M. We denote by $\mathrm{CL}^m(M,E)$ the classical (1-step polyhomogeneous) pseudodifferential operators of order m acting between the sections of E. It is well-known that the operator trace, which is defined on operators of order $m < -\dim M$, cannot be extended (regularized) to a trace on the whole algebra $\mathrm{CL}^\infty(M,E) = \bigcup_{m\in\mathbb{R}} \mathrm{CL}^m(M,E)$. In fact, for M^n connected and $n > 1$, up to a scalar multiple there is only one tracial functional on $\mathrm{CL}^\infty(M,E)$, and that functional vanishes on pseudodifferential operators of order $m < -\dim M$, cf. Wodzicki [14].

This picture changes drastically if one passes to 'pseudodifferential suspensions' of the algebra $\mathrm{CL}^\infty(M,E)$. It was shown by R.B. Melrose [11] that for a 'natural' pseudodifferential suspension $\Psi^\infty_{\mathrm{sus}}(M,E)$ of $\mathrm{CL}^\infty(M,E)$ the operator trace on operators in $\Psi^\infty_{\mathrm{sus}}(M,E)$ of order $m < -\dim M - 1$, can be extended by a canonical regularization procedure to a trace on the full algebra. He then used this regularized trace to 'lift' the spectrally defined η-invariant of Atiyah-Patodi-Singer [1] to an *η-homomorphism* from the algebraic K-theory group $K_1^{\mathrm{alg}}(\Psi^\infty_{\mathrm{sus}}(M,E))$ to $\mathbb{C}$. Furthermore, by means of the variation of his generalized η-invariant, Melrose defined the *divisor flow*

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between two invertibles of the algebra $\Psi^{\infty}_{\rm sus}(M,E)$ that are in the same component of the set of elliptic elements, and showed that it enjoys properties analogous to the spectral flow for self-adjoint elliptic operators.

Working with a slightly modified notion of pseudodifferential suspension, and for an arbitrary dimension $p \in \mathbb{N}$ of the parameter space, Lesch and Pflaum [9] generalized Melrose's trace regularization to the p-fold suspended pseudodifferential algebra $\mathrm{CL}^{\infty}(M,E;\mathbb{R}^p)$ of classical parameter dependent pseudodifferential operators. They also generalized Melrose's η-invariant to odd parametric dimensions, defining for $p = 2k+1$ the *higher η-invariant* $\eta_{2k+1}(A)$ of an invertible $A \in \mathrm{CL}^{\infty}(M,E;\mathbb{R}^p)$. The appellative 'eta' is justified by their result according to which any first-order invertible self-adjoint differential operator D can be canonically 'suspended' to an invertible parametric differential operator $\mathcal{D} \in \mathrm{CL}^1(M,E;\mathbb{R}^{2k+1})$, whose higher eta invariant $\eta_{2k+1}(\mathcal{D})$ coincides with the spectral η-invariant $\eta(D)$. On the negative side, in contrast with Melrose's η-homomorphism, the higher eta invariants are no longer additive on the multiplicative group of invertible elements. The 'defect of additivity' is purely symbolic though.

The starting point for the developments that make the object of the present exposition was the fundamental observation that the higher η-invariants η_{2k+1}, when assembled together with symbolic corrections into *higher divisor flows* DF_{2k+1}, can be understood as the expression of the Connes *pairing* between the topological K-theory of the pair $\big(\mathrm{CL}^0(M,E;\mathbb{R}^{2k+1}), \mathrm{CL}^{-\infty}(M,E;\mathbb{R}^{2k+1})\big)$ and a certain canonical *relative cyclic cocycle*, determined by the regularized *graded trace* together with its symbolic coboundary. The first such invariant, for $k = 0$, recovers Melrose's divisor flow, whose essential properties such as homotopy invariance, additivity and integrality, thus acquire a conceptual explanation. Of course, the same properties are shared by the higher divisor flows DF_{2k+1}. Furthermore, this interpretation allows to uncover the formerly 'missing' even dimensional higher eta invariants η_{2k}, with $k > 0$ and their associated divisor flows DF_{2k}. Taken collectively, the higher divisor flows $\mathrm{DF}_{\bullet}$ implement the natural Bott isomorphisms between the topological $K_{\bullet}$-groups of the pair $\big(\mathrm{CL}^0(M,E;\mathbb{R}^{\bullet}), \mathrm{CL}^{-\infty}(M,E;\mathbb{R}^{\bullet})\big)$ and $\mathbb{Z}$, in a manner compatible with the suspension isomorphisms in both K-theory and in cyclic cohomology. Finally, we clarify the relationship between the spectral flow and the higher divisor flows, by relating the latter to 'suspended' versions of the former, in all parametric dimensions.

The paper, which is of an expository nature, is written with the intent of presenting a clear and rather comprehensive description of the results, with precise references to the original sources for their proofs.

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1. Trace regularization and η-homomorphism

We start by recalling the definition of the p-fold suspended pseudodifferential algebra $\mathrm{CL}^m(M,E;\mathbb{R}^p)$ of classical parameter dependent pseudodifferential operators (cf. Lesch–Pflaum [9, Sec. 2]). It was originally defined by R.B. Melrose [11], in the case $p=1$, as a certain subalgebra of $\mathrm{CL}^\infty(M\times\mathbb{R},E)$ consisting of translation invariant operators. In our alternative set-up, Melrose's suspended algebra becomes a subalgebra of $\mathrm{CL}^\infty(M,E;\mathbb{R})$, after taking a partial Fourier transform in the $\mathbb{R}$-variable.

An element $A\in\mathrm{CL}^m(M,E;\mathbb{R}^p)$ is a map $\mathbb{R}^p\to\mathrm{CL}^m(M,E), \mu\mapsto A(\mu)$ which is locally given by

$$\begin{aligned}\big[\operatorname{Op}(a(\mu_0))\,u\big]\,(x) &:= \big[A(\mu_0)\,u\big](x)\\ &:= \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}\,a(x,\xi,\mu_0)\,\hat{u}(\xi)\,đ\xi\\ &= \int_{\mathbb{R}^n}\int_U e^{i\langle x-y,\xi\rangle}\,a(x,\xi,\mu_0)\,u(y)dyđ\xi.\end{aligned}\tag{1.1}$$

Here $a(x,\xi,\mu)$ is a classical symbol on $U\times(\mathbb{R}^n\times\mathbb{R}^p)$. More precisely, a is a smooth (matrix valued) function on $U\times\mathbb{R}^n\times\mathbb{R}^p$ such that for multi-indices $\alpha,\beta\in\mathbb{Z}^n_+,\gamma\in\mathbb{Z}^p_+$ and each compact subset $K\subset U$ we have an estimate

$$\begin{aligned}\big|\partial_x^\alpha\partial_\xi^\beta\partial_\mu^\gamma a(x,\xi,\mu)\big| \le C_{\alpha,\beta,\gamma,K}(1+|\xi|+|\mu|)^{m-|\beta|-|\gamma|},\\ x\in K,\xi\in\mathbb{R}^n,\mu\in\mathbb{R}^p.\end{aligned}\tag{1.2}$$

Furthermore, being classical means that a has an asymptotic expansion of the form

$$a\sim\sum_{j=0}^\infty a_j,\tag{1.3}$$

where $a_j\in\mathcal{C}^\infty(U\times\mathbb{R}^n\times\mathbb{R}^p)$ satisfies $a_j(x,\lambda\xi,\lambda\mu)=\lambda^{m-j}a_j(x,\xi,\mu)$ for $\lambda\ge1,|\xi|^2+|\mu|^2\ge1$.

In the case $p=0$ we obtain the usual (classical) pseudodifferential operators of order m on U. Parameter dependent pseudodifferential operators play a crucial role, e.g., in the construction of the resolvent expansion of an elliptic operator (Gilkey [5]). The definition of the parameter dependent calculus is not uniform in the literature. It will be crucial in the sequel that differentiating by the parameter reduces the order of the operator. This is the convention e.g. of [5] but differs from the one in Shubin [13].

On $\mathrm{CL}^m(M,E;\mathbb{R}^p)$, $m<-\dim M$, the operator trace induces a function valued trace

$$\mathrm{TR}(A)(\mu):=\mathrm{tr}_{L^2}(A(\mu)).\tag{1.4}$$

This function is integrable if $p+m<-\dim M$ and we obtain a trace on $\mathrm{CL}^m(M,E;\mathbb{R}^p)$ by putting

$$\overline{\mathrm{TR}}(A):=\int_{\mathbb{R}^p}\mathrm{tr}(A(\mu))d\mu.\tag{1.5}$$

Melrose showed in the case $p = 1$ that (1.5) can be regularized to give a trace on the whole algebra $\mathrm{CL}^\infty(M, E; \mathbb{R})$. More generally, for any $p \in \mathbb{N}$, it was shown in [9, Thm. 2.2, 4.6] that there exists a linear extension of (1.4), modulo polynomials, to operators of all orders

$$\mathrm{TR} : \mathrm{CL}^\infty(M, E; \mathbb{R}^p) \to \mathrm{PS}^\infty(\mathbb{R}^p)/\mathbb{C}[\mu_1, \dots, \mu_p],$$

which is uniquely determined by the following two properties:

(1) $\mathrm{TR}(AB) = \mathrm{TR}(BA)$, i.e. TR is tracial ;
(2) $\mathrm{TR}(\partial_j A) = \partial_j \mathrm{TR}(A)$ for $j = 1, ..., p$.

Furthermore, this unique extension TR satisfies

(3) $\mathrm{TR}(\mu_j A) = \mu_j \mathrm{TR}(A)$ for $j = 1, ..., p$.
(4) $\mathrm{TR}(\mathrm{CL}^m(M, E; \mathbb{R}^p)) \subset \mathrm{PS}^{m+p}(\mathbb{R}^p)/\mathbb{C}[\mu_1, \dots, \mu_p]$.

Here, $\mathrm{PS}^\infty(\mathbb{R}^p)$ is the class of functions on $\mathbb{R}^p$ having a complete asymptotic expansion in terms of homogeneous functions and log-powers as $\mu \to \infty$.

Composing any linear functional on $\mathrm{PS}^\infty(\mathbb{R}^p)/\mathbb{C}[\mu_1, \dots, \mu_p]$ with TR yields a trace on $\mathrm{CL}^\infty(M, E; \mathbb{R}^p)$. The *regularized trace*, that extends (1.5) and which still will be denoted by

$$\overline{\mathrm{TR}} : \mathrm{CL}^\infty(M, E; \mathbb{R}^p) \to \mathbb{C}\,, \tag{1.6}$$

is obtained by composition with the following regularization of the multiple integral. If $f \in \mathrm{PS}^m(\mathbb{R}^p)$ then

$$\int_{|\mu| \le R} f(\mu) d\mu \sim_{R\to\infty} \sum_{\alpha \to -\infty} p_{f,\alpha}(\log R) R^\alpha, \tag{1.7}$$

where $p_{f,\alpha}$ is a polynomial of degree $k(\alpha)$. The *regularized integral* of f is defined as the constant term in this asymptotic expansion:

$$⨍_{\mathbb{R}^p} f(\mu) d\mu \, := \, p_{f,0}(0). \tag{1.8}$$

For more details concerning the properties of this regularized integral, we refer to [7, Sec. 5] and [9].

In what follows, we shall view $\overline{\mathrm{TR}}$ as a linear functional on the space of differential forms on $\mathbb{R}^p$ with coefficients in $\mathrm{CL}^\infty(M, E; \mathbb{R}^p)$,

$$\overline{\mathrm{TR}} : \Omega^\bullet(\mathbb{R}^p, \mathrm{CL}^\infty(M, E; \mathbb{R}^p)) \to \mathbb{C}\,, \tag{1.9}$$

by just applying the extended trace to the coefficient of the volume form. This will be explained in more detail in the next section. At this point we would just like to emphasize that the functional thus defined is *not* a closed trace. Rather, for a $(p-1)$-form on $\mathbb{R}^p$ with coefficients in $\mathrm{CL}^\infty(M, E; \mathbb{R}^p)$, the *formal trace* is defined as

$$\widetilde{\mathrm{TR}}(\omega) := \overline{\mathrm{TR}}(d\omega)\,.$$

Although $\widetilde{\mathrm{TR}}$ is non-trivial, it was shown in [11] for $p = 1$, and in [9] in general, that it is *symbolic*, i.e. can be calculated (similarly to Wodzicki's noncommutative residue) by integrating a density which depends only on finitely many terms in the symbol expansion (1.3). Since the symbol expansion of a pseudodifferential operator is a *local* invariant some authors use the term 'local' instead of 'symbolic'.

Using this regularized trace, Melrose has defined in [11] the η-invariant of an invertible element $A \in \mathrm{GL}_1(\mathrm{CL}^\infty(M, E; \mathbb{R}))$ as the complex number

$$\eta(A) = \frac{-1}{\pi i} \overline{\mathrm{TR}}(A^{-1} dA) := \frac{-1}{\pi i} \,⨍_{\mathbb{R}} \mathrm{TR}\left(A^{-1} \frac{dA}{d\mu}\right) d\mu, \tag{1.10}$$

and showed that the assignment $A \in \mathrm{GL}(\mathrm{CL}^\infty(M, E; \mathbb{R})) \mapsto \eta(A)$ gives rise to a homomorphism $\eta : K_1^{\mathrm{alg}}(\mathrm{CL}^\infty(M, E; \mathbb{R})) \to \mathbb{C}$. He also showed that if

$$\mathcal{D}(\mu) := \partial\!\!\!/ + i\mu\,, \qquad \mu \in \mathbb{R}\,,$$

where $\partial\!\!\!/$ is the Dirac operator on a Riemannian spin manifold M, then

$$\eta(\mathcal{D}) = \eta(\partial\!\!\!/),$$

where the right hand side stands for the usual η-invariant of the Dirac operator, as defined by Atiyah-Patodi-Singer [1].

This was subsequently generalized by Lesch and Pflaum to odd dimensional parameter spaces as follows: if $A \in \mathrm{CL}(M, E; \mathbb{R}^{2k+1})$ is invertible then

$$\eta_{2k+1}(A) := \frac{2\,k!}{(-2\pi i)^{k+1}(2k+1)!} \overline{\mathrm{TR}}((A^{-1} dA)^{2k+1}) \tag{1.11}$$

(up to a sign) was called the (parametric) η-invariant of A. The terminology is justified by the fact that if D is an invertible first order self-adjoint elliptic differential operator and if $c : \mathbb{R}^{2k+1} \to \mathfrak{M}_{2^k}(\mathbb{C})$ is the standard Clifford representation then its $(2k+1)$*-fold suspension*

$$\mathcal{D}(\mu) := D + c(\mu)\,, \qquad \mu \in \mathbb{R}^{2k+1}\,, \tag{1.12}$$

is an invertible element of $\mathrm{CL}^\infty(M, E; \mathbb{R}^{2k+1})$ and the parametric η-invariant of $\mathcal{D}$ equals the *spectral* η-invariant of the operator D (cf. [11, Prop. 5] for Dirac operators and $k = 0$, [9, Prop. 6.6] in general).

It should be noted though that, unlike η_1, the higher η_{2k+1}, $k > 0$, are no longer additive on the multiplicative group of invertibles.

We now pause to clarify this important issue. To this end, assume that $A, B \in \mathrm{GL}_1(\mathrm{CL}^\infty(M, E; \mathbb{R}^{2k+1}))$ and consider the expression

$$\overline{\mathrm{TR}}\big((A^{-1} dA)^{2k+1}\big) \tag{1.13}$$

occurring in the definition of the divisor flow. First of all, when $k = 0$, we infer from

$$(AB)^{-1} d(AB) = B^{-1}\big(A^{-1} dA + dB B^{-1}\big) B \tag{1.14}$$

and the trace property of $\overline{\mathrm{TR}}$ the equality

$$\overline{\mathrm{TR}}\big((AB)^{-1} d(AB)\big) = \overline{\mathrm{TR}}(A^{-1} dA) + \overline{\mathrm{TR}}(B^{-1} dB), \tag{1.15}$$

showing that η_1 defines a homomorphism $\mathrm{GL}_1(\mathrm{CL}^\infty(M, E; \mathbb{R}^{2k+1})) \to \mathbb{C}$.

To understand the case $k \geq 1$ we need to recall the variation formula for the parametric η-invariant, cf. [11, Prop. 7] and [9, Prop. 6.3] . Let $A_s \in \mathrm{CL}^m(M,E;\mathbb{R}^{2k+1})$ be invertible and smoothly depending on a parameter s. Then

$$\frac{d}{ds}\eta_{2k+1}(A_s) = \frac{2\,k!}{(-2\pi i)^{k+1}(2k)!}\widetilde{\mathrm{TR}}\left((A_s^{-1}\partial_s A_s)(A_s^{-1}dA_s)^{2k}\right). \tag{1.16}$$

Consider now two invertible elements $a, b \in \mathrm{CL}^0(M,E;\mathbb{R}^{2k+1})$, for simplicity of order 0. Put

$$A := \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \quad B := \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \in \mathfrak{M}_2\left(\mathrm{CL}^0(M,E;\mathbb{R}^{2k+1})\right) \tag{1.17}$$

and

$$A_s := \begin{pmatrix} \cos(\frac{\pi}{2}s) & \sin(\frac{\pi}{2}s) \\ -\sin(\frac{\pi}{2}s) & \cos(\frac{\pi}{2}s) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} \cos(\frac{\pi}{2}s) & -\sin(\frac{\pi}{2}s) \\ \sin(\frac{\pi}{2}s) & \cos(\frac{\pi}{2}s) \end{pmatrix}. \tag{1.18}$$

A_s is a path of invertible elements of $\mathfrak{M}_2\left(\mathrm{CL}^0(M,E;\mathbb{R}^{2k+1})\right)$ with

$$A_1 = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \tag{1.19}$$

and A_0 satisfies

$$A_0 B = B A_0, \quad A_0 dB = (dB)A_0, \quad dA_0 \wedge dB = 0. \tag{1.20}$$

Consequently,

$$\left((A_0B)^{-1}d(A_0B)\right)^p = (A_0^{-1}dA_0)^p + (B^{-1}dB)^p, \quad p \in \mathbb{Z}_+, \tag{1.21}$$

and the variation formula (1.16) yields

$$\begin{aligned}
&\eta_{2k+1}(AB) - \eta_{2k+1}(A) - \eta_{2k+1}(B) \\
&= \eta_{2k+1}(AB) - \eta_{2k+1}(A) - \eta_{2k+1}(A_0B) + \eta_{2k+1}(A_0) \\
&= \frac{2\,k!}{(-2\pi i)^{k+1}(2k)!}\widetilde{\mathrm{TR}}\Big(\int_0^1 A_s^{-1}\partial_s A_s\big((A_s^{-1}dA_s + dBB^{-1})^{2k} - (A_s^{-1}dA_s)^{2k}\big)ds\Big).
\end{aligned} \tag{1.22}$$

Hence the defect of the additivity of η_{2k+1} is symbolic. Eq. (1.22) shows in fact that there is a noncommutative polynomial $P_{2k+1}(A,B,dA,dB)$ such that

$$\eta_{2k+1}(AB) - \eta_{2k+1}(A) - \eta_{2k+1}(B) = \widetilde{\mathrm{TR}}(P_{2k+1}(A,B,dA,dB)). \tag{1.23}$$

Thus, (1.22) holds for arbitrary invertible elements $A, B \in \mathrm{GL}_1(\mathrm{CL}^\infty(M,E;\mathbb{R}^{2k+1}))$, regardless of their orders.

As a concrete illustration, let us explicitly calculate the above polynomial in the case $k = 1$. Denoting $\omega_1 = B^{-1}A^{-1}dAB$ and $\omega_2 := B^{-1}dB$, one has

$$\begin{aligned}
&d\omega_1 = -\omega_1^2 - \omega_1 \wedge \omega_2 - \omega_2 \wedge \omega_1, \quad d\omega_2 = -\omega_2^2, \\
&d(\omega_1 + \omega_2) = -(\omega_1 + \omega_2)^2, \quad d(\omega_1 \wedge \omega_2) = -(\omega_1 + \omega_2) \wedge \omega_1 \wedge \omega_2.
\end{aligned} \tag{1.24}$$

These identities imply

$$
\begin{aligned}
&\overline{\mathrm{TR}}\Big(\big((AB)^{-1}d(AB)\big)^3\Big) - \overline{\mathrm{TR}}\big((A^{-1}dA)^3\big) - \overline{\mathrm{TR}}\big((B^{-1}dB)^3\big)\\
&\quad = \mathrm{TR}\big((\omega_1+\omega_2)^3\big) - \overline{\mathrm{TR}}\big(\omega_1^3\big) - \overline{\mathrm{TR}}\big(\omega_2^3\big)\\
&\quad = -3\,\mathrm{TR}\big(d(\omega_1\wedge\omega_2)\big)\\
&\quad = -3\widetilde{\mathrm{TR}}(B^{-1}A^{-1}dA\wedge dB),
\end{aligned}
\tag{1.25}
$$

and hence

$$
\eta_3(AB) = \eta_3(A) + \eta_3(B) - \frac{1}{4\pi^2}\widetilde{\mathrm{TR}}(B^{-1}A^{-1}dA\wedge dB). \tag{1.26}
$$

Returning to the variation formula for the η-invariant we note that the right hand side of (1.16) still makes sense if A_s is only elliptic and A_s^{-1} is replaced by a smooth family of parametrices Q_s. Thus, one can define the *divisor flow* $\mathrm{DF}_{2k+1}\big((A_s)_{0\le s\le 1}\big)$ by the equation

$$
\begin{aligned}
&\mathrm{DF}\,\big((A_s)_{0\le s\le 1}\big) =\\
&\quad = \frac{k!}{(-2\pi i)^{k+1}(2k+1)!}\Big(\overline{\mathrm{TR}}\,\big((A_1^{-1}dA_1)^{2k+1}\big) - \overline{\mathrm{TR}}\,\big((A_0^{-1}dA_0)^{2k+1}\big)\Big)\\
&\qquad - \frac{k!}{(-2\pi i)^{k+1}(2k)!}\int_0^1 \widetilde{\mathrm{TR}}\,\Big(Q_s\,\partial_s A_s\,\big(Q_s\,dA_s\big)^{2k}\Big)\,ds.
\end{aligned}
\tag{1.27}
$$

For $k=0$ this is precisely the divisor flow originally defined by Melrose [11, Eq. (5)], while for $k>0$ it gives its generalization by Lesch-Pflaum [9, Eq. (6.51)].

We note that, at this stage, the divisor flow is defined on *paths* of elliptic elements of $\mathrm{CL}^m(M,E;\mathbb{R}^p)$ with invertible endpoints. The fact that it actually depends only on the *homotopy classes* of such paths, as well as its other key properties, such as *additivity* and *integrality*, were fully established in [8], along the lines which we proceed now to explain.

2. Divisor flow as relative cyclic pairing

As a first step towards a better understanding of the divisor flow, we shall recast its definition – and at the same time generalize it – in the framework of noncommutative geometry.

2.1. **Relative cyclic pairing.** We start by formulating the definition of the natural pairing between relative cyclic homology and cohomology in the 'mapping cone' setup, that is best suited to the purposes of this paper.

First, recall (cf. [2], [10]) that to every unital $\mathbb{C}$-algebra $\mathcal{A}$ (possibly endowed with a locally convex topology) one can associate the mixed complex

$(C^\bullet(\mathcal{A}), b, B)$, where $C^k(\mathcal{A}) = \big(\mathcal{A}\otimes\mathcal{A}^{\otimes k}\big)^*$,

$$b\phi(a_0,\dots,a_{k+1}) = \sum_{j=0}^{k}(-1)^j\,\phi(a_0,\dots,a_ja_{j+1},\dots,a_{k+1}) + (-1)^{k+1}\phi(a_{k+1}a_0,a_1,\dots,a_k),$$

and

$$B\phi(a_0,\dots,a_{k-1}) = \sum_{j=0}^{k-1}(-1)^{(k-1)j}\,\phi(1,a_j,\dots,a_{k-1},a_0,\dots,a_{j-1}) - \sum_{j=0}^{k-1}(-1)^{(k-1)j}\,\phi(a_j,1,a_{j+1},\dots,a_k,a_0,\dots,a_{j-1}).$$

One can then form the double complexes $\mathcal{B}C^{\bullet,\bullet}(\mathcal{A})$ and $\mathcal{B}C^{\bullet,\bullet}_{\mathrm{per}}(\mathcal{A})$. Their (nonvanishing) components are defined as $\mathcal{B}C^{p,q}(\mathcal{A}) = C^{q-p}(\mathcal{A})$ for $q\geq p\geq 0$ resp. $\mathcal{B}C^{p,q}_{\mathrm{per}}(\mathcal{A}) = C^{q-p}(\mathcal{A})$ for $q\geq p$, and have B as horizontal, resp. b as vertical differential. The *cyclic* resp. *periodic cyclic cohomology* groups of $\mathcal{A}$ are obtained as follows:

$$HC^\bullet(\mathcal{A}) = H^\bullet(\mathrm{Tot}^\bullet_\bigoplus \mathcal{B}C^{\bullet,\bullet}(\mathcal{A})) \ \text{ resp. } \ HP^\bullet(\mathcal{A}) = H^\bullet(\mathrm{Tot}^\bullet_\bigoplus \mathcal{B}C^{\bullet,\bullet}_{\mathrm{per}}(\mathcal{A}));$$

in both cases the total differential is $b+B$.

Let now

$$0\longrightarrow\mathcal{J}\longrightarrow\mathcal{A}\xrightarrow{\sigma}\mathcal{B}\longrightarrow 0. \tag{2.1}$$

be a short exact sequence of unital (Fréchet) algebras and (continuous) homomorphisms. Consider the associated morphism of mixed complexes

$$\sigma^*: C^\bullet(\mathcal{B})\to C^\bullet(\mathcal{A}),$$

and form the corresponding mapping cone of total complexes

$$\big(\mathrm{Tot}^\bullet_\oplus \mathcal{B}C^{\bullet,\bullet}(\mathcal{A})\oplus\mathrm{Tot}^{\bullet+1}_\oplus \mathcal{B}C^{\bullet,\bullet}(\mathcal{B}), \widetilde{b+B}\big),$$

with the differential

$$\widetilde{b+B} = \begin{pmatrix} b+B & -\sigma^* \\ 0 & -(b+B)\end{pmatrix}.$$

Explicitly, $\mathrm{Tot}^k_\oplus \mathcal{B}C^{\bullet,\bullet}(\mathcal{A})\oplus\mathrm{Tot}^{k+1}_\oplus \mathcal{B}C^{\bullet,\bullet}(\mathcal{B})\cong$

$$\cong\bigoplus_{p+q=k}\mathcal{B}C^{p,q}(\mathcal{A})\oplus\mathcal{B}C^{p,q+1}(\mathcal{B}) = \mathrm{Tot}^k_\oplus \mathcal{B}C^{\bullet,\bullet}(\mathcal{A},\mathcal{B}),$$

where $\mathcal{B}C^{\bullet,\bullet}(\mathcal{A},\mathcal{B})$ is the double complex associated to the relative mixed complex $(C^\bullet(\mathcal{A},\mathcal{B}),\widetilde{b},\widetilde{B})$, which is given by $C^k(\mathcal{A},\mathcal{B}) = C^k(\mathcal{A})\oplus C^{k+1}(\mathcal{B})$,

$$\widetilde{b} = \begin{pmatrix} b & -\sigma^* \\ 0 & -b\end{pmatrix}, \ \text{ and } \ \widetilde{B} = \begin{pmatrix} B & 0 \\ 0 & -B\end{pmatrix}.$$

Hence the *relative cyclic cohomology* $HC^\bullet(\mathcal{A},\mathcal{B})$, resp. the *relative periodic cyclic cohomology* $HP^\bullet(\mathcal{A},\mathcal{B})$ can be realized as the cohomology of

$$\big(\mathrm{Tot}^\bullet_\oplus \mathcal{B}C^{\bullet,\bullet}(\mathcal{A},\mathcal{B}),\widetilde{b}+\widetilde{B}\big) \quad\text{resp.}\quad \big(\mathrm{Tot}^\bullet_\oplus \mathcal{B}C^{\bullet,\bullet}_{\mathrm{per}}(\mathcal{A},\mathcal{B}),\widetilde{b}+\widetilde{B}\big).$$

The preceding constructions can be dualized in an obvious fashion. Thus, $HC_\bullet(\mathcal{A},\mathcal{B})$ is the homology of $\left(\mathrm{Tot}_\bullet^{\oplus}\,\mathcal{B}C_{\bullet,\bullet}(\mathcal{A},\mathcal{B}), \widetilde{b}+\widetilde{B}\right)$, where $\mathcal{B}C_{p,q}(\mathcal{A},\mathcal{B}) = \mathcal{B}C_{p,q}(\mathcal{A}) \oplus \mathcal{B}C_{p,q+1}(\mathcal{B})$,

$$\widetilde{b} = \begin{pmatrix} b & 0 \\ -\sigma_* & -b \end{pmatrix}, \text{ and } \widetilde{B} = \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix}.$$

Likewise, the periodic cyclic homology $HP_\bullet(\mathcal{A},\mathcal{B})$ is the homology of $\left(\mathrm{Tot}_\bullet^{\prod}\,\mathcal{B}C^{\mathrm{per}}_{\bullet,\bullet}(\mathcal{A},\mathcal{B}), \widetilde{b}+\widetilde{B}\right)$, where $\mathcal{B}C^{\mathrm{per}}_{p,q}(\mathcal{A},\mathcal{B}) = \mathcal{B}C^{\mathrm{per}}_{p,q}(\mathcal{A}) \oplus \mathcal{B}C^{\mathrm{per}}_{p,q+1}(\mathcal{B})$ and where $\widetilde{b}$, $\widetilde{B}$ are as above.

The pairs of dual complexes thus obtained inherit a natural pairing:

$$\langle(\varphi_\bullet,\psi_{\bullet+1}),(a_\bullet,b_{\bullet+1})\rangle\rangle \mapsto \langle\varphi_\bullet,a_\bullet\rangle + \langle\psi_{\bullet+1},b_{\bullet+1}\rangle \tag{2.2}$$

Next, we recall that the notion of a cycle, introduced by Connes in [2], has a natural extension to the relative situation.

Definition 2.1 (cf. [6, Sec. 2],[8, Def. 1.9])**.** A *relative cycle* of degree k over the pair of algebras $(\mathcal{A},\mathcal{B})$ consists of the following data:

(1) differential graded unital algebras $(\Omega^\bullet, d)$ and $(\partial\Omega^\bullet, d)$ over $\mathcal{A}$ resp. $\mathcal{B}$, together with a surjective unital homomorphism $r:\Omega^\bullet\to\partial\Omega^\bullet$ of degree 0;
(2) unital homomorphisms $\varrho_{\mathcal{A}}:\mathcal{A}\to\Omega^0$ and $\varrho_{\mathcal{B}}:\mathcal{B}\to\partial\Omega^0$ such that $r\circ\varrho_{\mathcal{A}}=\varrho_{\mathcal{B}}\circ\sigma$;
(3) a graded trace $\int$ on $\Omega^\bullet$ of degree k such that

$$\int d\omega = 0\,, \quad \text{whenever} \quad r(\omega)=0. \tag{2.3}$$

Given a relative cycle $C=\left(\Omega^\bullet,\partial\Omega^\bullet,r,\int,\int'\right)$ of degree p over $(\mathcal{A},\mathcal{B})$, we define the cochain $(\varphi_p,\psi_{p-1})\in C^p(\mathcal{A})\oplus C^{p-1}(\mathcal{B})$ by

$$\varphi_p(a_0,\ldots,a_p) := \frac{1}{p!}\int \varrho(a_0)d\varrho(a_1)\ldots d\varrho(a_p), \tag{2.4}$$

$$\psi_{p-1}(b_0,\ldots,b_{p-1}) := \frac{1}{(p-1)!}\int' \varrho(b_0)d\varrho(b_1)\ldots d\varrho(b_{p-1}). \tag{2.5}$$

It is straightforward to check (cf. [6, Sec. 2], [8, Prop. 1.10]) that the pair

$$\mathrm{char}\, C := (\varphi_p,\psi_{p-1}) \in \mathrm{Tot}^p_{\oplus}\,\mathcal{B}C^{\bullet,\bullet}(\mathcal{A},\mathcal{B})$$

is a relative cyclic cocycle, called the *character* of the relative cycle C.

We are now in a position to define the notion of divisor flow in the general framework of noncommutative geometry. Let $(a_s)_{0\le s\le 1}$ be a smooth *admissible elliptic* path $(a_s)_{0\le s\le1}$ of elements in $\mathfrak{M}_N(\mathcal{A})$ for some $N\in\mathbb{N}$; *elliptic* means here that $\sigma(a_s)$ is invertible in $\mathfrak{M}_N(\mathcal{B})$ for each $s\in[0,1]$, while *admissible* means that a_0, a_1 are both in $\mathrm{GL}_N(\mathcal{A})$. Such a path gives rise to a relative cyclic homology class, namely

$$\begin{aligned} &\mathrm{ch}_\bullet\big((a_s)_{0\le s\le 1}\big) \\ &\quad := \Big(\mathrm{ch}_\bullet(a_1)-\mathrm{ch}_\bullet(a_0), -\int_0^1 \not{c}\mathrm{h}_\bullet\left(\sigma(a_s),\sigma(\dot a_s)\right) ds\Big). \end{aligned} \tag{2.6}$$

Here, $\mathrm{ch}_\bullet(g)$ stands for the odd Chern character of an invertible $g\in\mathrm{GL}_\infty(\mathcal{A})$,

$$\mathrm{ch}_\bullet(g)=\sum_{k=0}^{\infty}(-1)^k\, k!\ \mathrm{tr}_{2k+1}\left((g^{-1}\otimes g)^{\otimes k}\right), \tag{2.7}$$

while $\not{c}\mathrm{h}_\bullet(h,\dot h)$, with $h_s\in\mathrm{GL}_\infty(\mathcal{B})$, $s\in[0,1]$, denotes its secondary Chern character (see [4]):

$$\begin{aligned} &\not{c}\mathrm{h}_\bullet(h,\dot h)=\mathrm{tr}_0(h^{-1}\dot h)+ \\ &\quad+\sum_{k=0}^{\infty}(-1)^{k+1}k!\sum_{j=0}^{k}\mathrm{tr}_{2k+2}\left((h^{-1}\otimes h)^{\otimes(j+1)}\otimes h^{-1}\dot h\otimes(h^{-1}\otimes h)^{\otimes(k-j)}\right). \end{aligned}$$

The known transgression formula for the odd Chern character,

$$\frac{d}{ds}\,\mathrm{ch}_\bullet(h)=(b+B)\ \not{c}\mathrm{h}_\bullet(h,\dot h),$$

ensures that Eq. (2.6) does define a relative cyclic cycle. When properly interpreted (cf. Theorem 3.1 below), it will turn out to be the *Chern character* in relative cyclic homology of the relative K-theory class defined by the $(a_s)_{0\le s\le 1}$.

Given a short exact sequence of Fréchet algebras of the form (2.1), let C be an odd relative cycle with character $(\varphi_{2k+1},\psi_{2k})$. The (*odd*) *divisor flow* with respect to C of a smooth admissible elliptic path $(a_s)_{0\le s\le 1}$ is the relative pairing between $\mathrm{ch}_\bullet\left((a_s)_{0\le s\le 1}\right)$ and the character $\mathrm{char}\, C=(\varphi_{2k+1},\psi_{2k})$:

$$\begin{aligned} \mathrm{DF}_C\left((a_s)_{0\le s\le 1}\right) &:= \mathrm{DF}\left((a_s)_{0\le s\le 1}\right) := \\ &:= \frac{1}{(-2\pi i)^{k+1}}\langle \mathrm{char}\, C,\ \mathrm{ch}_\bullet((a_s)_{0\le s\le 1})\rangle \\ &= \frac{1}{(-2\pi i)^{k+1}}\left(\langle\varphi_{2k+1},\mathrm{ch}_\bullet(a_1)\rangle-\langle\varphi_{2k+1},\mathrm{ch}_\bullet(a_0)\rangle\right) \\ &\quad -\frac{1}{(-2\pi i)^{k+1}}\langle\psi_{2k},\int_0^1\not{c}\mathrm{h}_\bullet(\sigma(a_s),\sigma(\dot a_s))ds\rangle. \end{aligned} \tag{2.8}$$

Simple calculations show that the partial pairings involved in the above formula can be expressed as follows:

$$\langle\varphi_{2k+1},\mathrm{ch}_\bullet(a_s)\rangle=\frac{k!}{(2k+1)!}\int\left(a_s^{-1}da_s\right)^{2k+1}, \tag{2.9}$$

$$\begin{aligned} &\langle\psi_{2k},\ \not{c}\mathrm{h}_\bullet(\sigma(a_s),\sigma(\dot a_s))\rangle \\ &\quad=\frac{k!}{(2k)!}\int'(\sigma(a_s)^{-1}\sigma(\dot a_s))\left((\sigma(a_s))^{-1}d(\sigma(a_s))\right)^{2k}. \end{aligned} \tag{2.10}$$

Turning to the even case, we recall that the Chern character of an idempotent $e \in \mathrm{P}_\infty(\mathcal{A})$ is given by the formula

$$\mathrm{ch}_\bullet(e) := 1 + \sum_{k=1}^{\infty} (-1)^k \frac{(2k)!}{k!} \operatorname{tr}_{2k} \Big(\big(e - \frac{1}{2}\big) \otimes e^{\otimes(2k)} \Big). \tag{2.11}$$

If $(e_s)_{0\le s\le 1}$ is a smooth path of idempotents, the corresponding transgression formula reads

$$\frac{d}{ds}\,\mathrm{ch}_\bullet(e_s) = (b+B)\,\not{c}\mathrm{h}_\bullet(e_s, (2e_s - 1)\dot{e}_s);$$

here the secondary Chern character $\not{c}\mathrm{h}_\bullet$ is given by

$$\not{c}\mathrm{h}_\bullet(e,h) := \iota(h)\,\mathrm{ch}_\bullet(e),$$

where

$$\iota(h)(a_0 \otimes a_1 \otimes \ldots \otimes a_l) = \sum_{i=0}^{l} (-1)^i (a_0 \otimes \ldots \otimes a_i \otimes h \otimes a_{i+1} \otimes \ldots \otimes a_l).$$

We now consider a smooth path of *almost idempotents* $(f_s)_{0\le s\le 1}$ in $\mathfrak{M}_N(\mathcal{A})$, i.e. such that $\sigma(f_s)$ are idempotents in $\mathfrak{M}_N(\mathcal{B})$, which is *admissible* in the sense that the endpoints are idempotents. By analogy with Eq. (2.6), we define its Chern character by

$$\begin{aligned} &\mathrm{ch}_\bullet\big((f_s)_{0\le s\le 1}\big) \\ &\quad := \Big(\mathrm{ch}_\bullet(f_1) - \mathrm{ch}_\bullet(f_0), -\int_0^1 \not{c}\mathrm{h}_\bullet\left(\sigma(f_s), \sigma((2f_s-1)\dot{f}_s)\right) ds\Big). \end{aligned} \tag{2.12}$$

In view of the above transgression formula, this expression gives a relative cyclic cycle. The relative pairing between $\mathrm{ch}_\bullet\big((f_s)_{0\le s\le 1}\big)$ and the character $\operatorname{char} C = (\varphi_{2k}, \psi_{2k-1})$ of an even relative cycle C defines the corresponding *even divisor flow*

$$\begin{aligned} \mathrm{DF}_C\big((f_s)_{0\le s\le 1}\big) &:= \mathrm{DF}\big((f_s)_{0\le s\le 1}\big) := \\ &:= \frac{(-1)^{k+1}}{(2\pi i)^k} \big\langle \operatorname{char} C,\, \mathrm{ch}_\bullet((f_s)_{0\le s\le 1})\big\rangle \\ &= \frac{(-1)^{k+1}}{(2\pi i)^k} \Big(\big\langle \varphi_{2k}, \mathrm{ch}_\bullet(f_1)\big\rangle - \big\langle \varphi_{2k}, \mathrm{ch}_\bullet(f_0)\big\rangle \Big) \\ &\quad + \frac{(-1)^k}{(2\pi i)^k} \big\langle \psi_{2k-1}, \int_0^1 \not{c}\mathrm{h}_\bullet(\sigma(f_s), \sigma((2f_s-1)\dot{f}_s)) ds \big\rangle. \end{aligned} \tag{2.13}$$

Explicitly, the partial pairings entering in the above formula are given by

$$\langle \varphi_{2k}, \mathrm{ch}_\bullet(f_s)\rangle = \frac{(-1)^k}{k!} \int \big(f_s - \frac{1}{2}\big)(df_s)^{2k}, \tag{2.14}$$

$$\begin{aligned} &\langle \psi_{2k-1}, \not{c}\mathrm{h}_\bullet(\sigma(f_s), \sigma((2f_s-1)\dot{f}_s))\rangle = \\ &\quad = \frac{(-1)^k}{(k-1)!} \int' \sigma((2f_s-1)\dot{f}_s)\big(d(\sigma(f_s))\big)^{2k-1}. \end{aligned} \tag{2.15}$$

2.2. **Relative pairing for de Rham classes.** To give a quick illustration of the preceding concepts in a familiar setting, we digress to consider the case of the Poincaré pairing on a compact n-dimensional manifold M with boundary ∂M.

Let $i : \partial M \hookrightarrow M$ denote the inclusion and let $\omega \in \Omega^{n-p}(M)$ be a fixed closed differential form. Set

$$\begin{aligned}
&\Omega := \Omega^\bullet(M), \quad \partial\Omega := \Omega^\bullet(\partial M), \quad r := i^*, \\
&\varrho : \mathcal{C}^\infty(M) \hookrightarrow \Omega^0(M), \ \mathcal{C}^\infty(\partial M) \hookrightarrow \Omega^0(\partial M), \\
&\int : \Omega^{n-p} \to \mathbb{C}, \ \int \eta := \int_M \eta \wedge \omega, \\
&\int' : \partial\Omega^{n-p-1} \to \mathbb{C}, \ \int' \eta := \int_{\partial M} \eta \wedge i^*\omega.
\end{aligned} \tag{2.16}$$

Then $C := C_\omega := \left(\Omega^\bullet, \partial\Omega^\bullet, r, \varrho, \int, \int'\right)$ is a relative cycle of degree k over

$$(\mathcal{A}, \mathcal{B}) := (\mathcal{C}^\infty(M), \mathcal{C}^\infty(\partial M)),$$

with character is (φ_p, ψ_{p-1}) given by

$$\begin{aligned}
\varphi_p(a_0, \ldots, a_p) &= \frac{1}{p!} \int_M a_0\, da_1 \wedge \ldots \wedge da_p \wedge \omega, \\
\psi_{p-1}(b_0, \ldots, b_{p-1}) &= \frac{1}{(p-1)!} \int_{\partial M} b_0\, db_1 \wedge \ldots \wedge db_{p-1} \wedge i^*\omega;
\end{aligned}$$

this is a relative cyclic cycle, whose periodic cyclic cohomology class in $HP^\bullet(\mathcal{C}^\infty(M), \mathcal{C}^\infty(\partial M))$ corresponds, via the Connes-type isomorphism

$$HP^\bullet(\mathcal{C}^\infty(M), \mathcal{C}^\infty(\partial M)) \simeq H_\bullet(M, \partial M),$$

to the class of the current defined by ω.

To fix the ideas, we shall assume that the cycle has degree $p = 2k+1$ or, equivalently, that ω has degree $n - 2k - 1$. If $g \in \mathrm{GL}_N(\mathcal{A})$, i.e. if g is a smooth map from M into $\mathrm{GL}_N(\mathbb{C})$, then g represents an element of $K^1(M) = K_1(\mathcal{A})$. Denote by $\mathrm{Ch}_{\mathrm{DG}}(g) \in \Omega^{\mathrm{odd}}(M)$ the form representing the classical Chern character (in differential geometry) of this element in $K^1(M)$; explicitly, cf. e.g. [4, Prop. 1.4],

$$\mathrm{Ch}_{\mathrm{DG}}(g) = \sum_{k=0}^\infty \frac{k!}{(-2\pi i)^{k+1}(2k+1)!} \operatorname{tr}(g^{-1}dg)^{2k+1}. \tag{2.17}$$

Then

$$\begin{aligned}
\langle \varphi_{2k+1}, \mathrm{ch}_\bullet(g)\rangle &= \frac{k!}{(2k+1)!} \int_M \operatorname{tr}(g^{-1}dg)^{2k+1} \wedge \omega \\
&= (-2\pi i)^{k+1} \int_M \mathrm{Ch}_{\mathrm{DG}}(g) \wedge \omega.
\end{aligned} \tag{2.18}$$

Apparently, the right hand side is not well-defined at the cohomological level, as both $\mathrm{Ch}_{\mathrm{DG}}(g)$ and ω represent absolute cohomology classes. However, let us consider a smooth admissible elliptic path $(g_s)_{0\leq s\leq 1}$ in $\mathfrak{M}_N(\mathcal{A})$ with $g_0 = I$. Admissibility means here that g_1 is invertible and that $g_s|\partial M$ is invertible for all s. Thus, $(g_s)_{0\leq s\leq 1}$ represents an element in the relative K^1-group

$$K^1(M,\partial M) \simeq K_1(\mathcal{C}^\infty(M), \mathcal{C}^\infty(\partial M))$$

and its divisor flow with respect to the relative cycle $C = C_\omega$ is given by

$$\begin{aligned}\mathrm{DF}_C\left((g_s)_{0\leq s\leq 1}\right) &= \frac{k!}{(-2\pi i)^{k+1}(2k+1)!}\int_M \mathrm{tr}(g_1^{-1}dg_1)^{2k+1}\wedge\omega - \\ &- \frac{k!}{(-2\pi i)^{k+1}(2k)!}\int_0^1\int_{\partial M} g_s^{-1}\dot{g}_s(g_s^{-1}dg_s)^{2k}\wedge i^*\omega ds,\end{aligned} \tag{2.19}$$

which is a well-defined invariant of the K-theory class $[(g_s)_{0\leq s\leq 1}] \in K^1(M,\partial M)$. Moreover, this invariant can be identified with the result of the Poincaré pairing between the classical Chern character of the above K^1-theory class, which belongs to $H^{\mathrm{odd}}(M,\partial M)$, and the absolute cohomology class $[\omega] \in H^\bullet(M)$. Indeed, by a standard argument in K-theory, one can lift $g_s|\partial M$ to an invertible element in $\mathfrak{M}_N(\mathcal{A})$ and represent the class of $(g_s)_{0\leq s\leq 1}$ in $K^1(M,\partial M)$ by an admissible path $(h_s)_{0\leq s\leq 1}$ with $h_0 = I, h_s|\partial M = I$. Then (2.19) reduces to

$$\begin{aligned}\mathrm{DF}\left((h_s)_{0\leq s\leq 1}\right) &= \frac{k!}{(-2\pi i)^{k+1}(2k+1)!}\int_M \mathrm{tr}(h_1^{-1}dh_1)^{2k+1}\wedge\omega \\ &= \int_M \mathrm{Ch}_{\mathrm{DG}}(h_1)\wedge\omega\,,\end{aligned} \tag{2.20}$$

which is precisely the usual Poincaré pairing between the relative de Rham cohomology class of $\mathrm{Ch}_{\mathrm{DG}}(h_1)$ and the absolute cohomology class of ω.

2.3. **Divisor flows of parametric pseudodifferential operators.** In order to interpret Melrose's divisor flow and its higher analogues within the above setup, we need to specify a relative cycle for the short exact sequence of parametric pseudodifferential operators and symbols introduced in the first section,

$$0 \longrightarrow \mathrm{CL}^{-\infty}(M,E;\mathbb{R}^p) \longrightarrow \mathrm{CL}^{\infty}(M,E;\mathbb{R}^p) \xrightarrow{\sigma} \mathrm{C}\mathcal{S}^{\infty}(M,E;\mathbb{R}^p) \longrightarrow 0\,. \tag{2.21}$$

Let $\Lambda^\bullet := \Lambda^\bullet(\mathbb{R}^p)^* = \mathbb{C}[d\mu_1,\ldots,d\mu_p]$ be the exterior algebra of the vector space $(\mathbb{R}^p)^*$. Put

$$\begin{aligned}\Omega_p &:= \mathrm{CL}^\infty(M,E;\mathbb{R}^p)\otimes\Lambda^\bullet \quad \text{and} \\ \partial\Omega_p &:= \mathrm{CL}^\infty(M,E;\mathbb{R}^p)/\mathrm{CL}^{-\infty}(M,E;\mathbb{R}^p)\otimes\Lambda^\bullet.\end{aligned} \tag{2.22}$$

Recall that the *regularized trace* of a k-form $A(\mu)d\mu_1\wedge\ldots\wedge d\mu_k$ is, by definition, equal to 0 if $k\neq p$, while

$$\overline{\mathrm{TR}}_p(A(\mu)d\mu_1\wedge\ldots\wedge d\mu_p) := \int\!\!\!\!\!\!-_{\mathbb{R}^p} \mathrm{TR}(A)(\mu)d\mu_1\wedge\ldots\wedge d\mu_p. \tag{2.23}$$

By construction, $\overline{\mathrm{TR}}_p$ is a graded trace on the differential algebra $(\Omega_p,\, d)$, but in general is not closed. However, its boundary,

$$\widetilde{\mathrm{TR}}_p := d \circ \overline{\mathrm{TR}}_p = \overline{\mathrm{TR}}_p \circ d\,,$$

called the *formal trace*, is a closed graded trace of degree $p-1$. It was shown in [9, Prop. 5.8], [11, Prop. 6] that $\widetilde{\mathrm{TR}}_p$ is *symbolic*, meaning that it descends to a well-defined closed graded trace of degree $p-1$ on $\partial\Omega_p^\bullet$. Together with the natural quotient map $r : \Omega_p^\bullet \to \partial\Omega_p^\bullet$ we have thus constructed a relative cycle in the sense of Definition 2.1:

$$C^p_{\mathrm{reg}} := (\Omega_p,\, \partial\Omega_p,\, r,\, \varrho,\, \overline{\mathrm{TR}}_p,\, \widetilde{\mathrm{TR}}_p)\,. \tag{2.24}$$

Note that for $p = 2k+1$, and with $A_s \in \mathrm{CL}^\infty(M, E; \mathbb{R}^{2k+1})$ a smooth family of elliptic operators of some fixed order m such that A_0 and A_1 are invertible, the pairing of $\mathrm{ch}_\bullet\big((A_s)_{0\le s\le 1}\big)$ with the character $(\varphi_{2k+1}, \psi_{2k})$ of the relative cycle C^{2k+1}_{reg} gives precisely the expression Eq. (1.27) of the higher divisor flows introduced in [9]. In particular, when $p = 1$ one recovers the original divisor flow of Melrose.

3. K-theoretical interpretation of the divisor flow

As already mentioned, the divisor flow can be best understood in the framework of relative K-theory and its pairing with cyclic cohomology. In order to be able to formulate it in this manner, we first need to establish the homotopy invariance of the divisor flow.

To this end, under the same general assumptions as in the previous section, let us consider a smooth family of matrices $(a_{s,t})_{0\le s,t\le 1}$ over the algebra $\mathcal{A}$, such that for each fixed t the family $(a_{s,t})_{0\le s\le 1}$ is a smooth admissible path. For every smooth two-parameter family of invertibles $g_{s,t} \in \mathrm{GL}_\infty(\mathcal{A})$, $s, t \in [0,1]$, there is a secondary transgression formula, which has the form

$$\frac{\partial}{\partial s}\, \not{c}\mathrm{h}_\bullet(g, \partial_t g) - \frac{\partial}{\partial t}\, \not{c}\mathrm{h}_\bullet(g, \partial_s g) = (b+B)\, \not{\not{c}}\mathrm{h}_\bullet(g, \partial_s g, \partial_t g), \tag{3.1}$$

where $\not{\not{c}}\mathrm{h}_\bullet$ stands for the ‘tertiary’ Chern character (see [8, Eq. (1.15)]). By applying it, one obtains that $\mathrm{ch}_\bullet\big((a_{s,1})_{0\le s\le 1}\big)$ and $\mathrm{ch}_\bullet\big((a_{s,0})_{0\le s\le 1}\big)$ are homologous relative cyclic cycles:

$$\begin{aligned}
&\mathrm{ch}_\bullet\big((a_{s,1})_{0\le s\le 1}\big) - \mathrm{ch}_\bullet\big((a_{s,0})_{0\le s\le 1}\big) \\
&\quad = \big(\widetilde{b} + \widetilde{B}\big)\Big(\not{c}\mathrm{h}\,\big(a_{s,1}, \partial_t a_{s,1}\big) - \not{c}\mathrm{h}\,\big(a_{s,0}, \partial_t a_{s,0}\big), \\
&\qquad\qquad - \int_0^1 \not{\not{c}}\mathrm{h}\,\big(\sigma(a_{s,t}), \sigma(\partial_s a_{s,t}), \sigma(\partial_t a_{s,t})\big) ds\Big).
\end{aligned}$$

This guarantees that the (Chern) character $\mathrm{ch}_\bullet$ descends to a map

$$\mathrm{ch}_\bullet : \pi_1\big(\mathrm{Ell}_\infty(\mathcal{A}), \mathrm{GL}_\infty(\mathcal{A})\big) \to HC_{\mathrm{odd}}(\mathcal{A}, \mathcal{B})\,, \tag{3.2}$$

where $\pi_1\big(\mathrm{Ell}_\infty(\mathcal{A}), \mathrm{GL}_\infty(\mathcal{A})\big)$ is the groupoid (with respect to concatenation) of all homotopy classes of smooth admissible elliptic paths.

There is also a related monoid structure on $\pi_1\big(\operatorname{Ell}_\infty(\mathcal{A}), \operatorname{GL}_\infty(\mathcal{A})\big)$ induced by pointwise multiplication, which becomes a group structure modulo the submonoid of null-homotopic paths. The corresponding quotient $\tilde{\pi}_1\big(\operatorname{Ell}_\infty(\mathcal{A}), \operatorname{GL}_\infty(\mathcal{A})\big)$ can be naturally identified with the subset $\pi_1\big(\operatorname{Ell}_\infty(\mathcal{A}), \operatorname{GL}_\infty(\mathcal{A}); I\big) \subset \pi_1\big(\operatorname{Ell}_\infty(\mathcal{A}), \operatorname{GL}_\infty(\mathcal{A})\big)$ of all homotopy classes of smooth admissible elliptic paths starting at the identity. The latter gives an alternate description of the relative K_1-group. Furthermore, the 'descended' character map (3.2) can be naturally identified with the standard Chern character.

Theorem 3.1 ([8, Thms. 1.6, 1.7])**.** *Assume that $\mathcal{A}$ and $\mathcal{B} = \mathcal{A}/\mathcal{J}$ are local Banach algebras. Then the relative K-theory group $K_1(\mathcal{A}, \mathcal{B})$ can be canonically identified with the group $\pi_1\big(\operatorname{Ell}_\infty(\mathcal{A}), \operatorname{GL}_\infty(\mathcal{A}); I\big)$ and the (periodized) inherited character map*

$$\operatorname{ch}_\bullet : K_1(\mathcal{A}, \mathcal{B}) \to HP_{\mathrm{odd}}(\mathcal{A}, \mathcal{B})$$

coincides, via the canonical identification $K_1(\mathcal{J}) \cong K_1(\mathcal{A}, \mathcal{B})$ with the standard Chern character in cyclic homology.

There is a parallel alternative description of the relative K_0-group of a pair of algebras $(\mathcal{A}, \mathcal{B})$ as above. Let $\Omega(\mathrm{AP}_\infty(\mathcal{A}), \mathrm{P}_\infty(\mathcal{A}))$ be the set of continuous paths of *almost idempotents*, i.e. matrices over $\mathcal{A}$ which are idempotent modulo $\mathcal{J}$, with endpoints in $\mathrm{P}_\infty(\mathcal{A})$. The direct sum of matrices turns $\Omega(\mathrm{AP}_\infty(\mathcal{A}), \mathrm{P}_\infty(\mathcal{A}))$, and also the set of homotopy classes $\pi_1(\mathrm{AP}_\infty(\mathcal{A}), \mathrm{P}_\infty(\mathcal{A}))$, into a monoid. A path $\gamma \in \Omega(\mathrm{AP}_\infty(\mathcal{A}), \mathrm{P}_\infty(\mathcal{A}))$ is called *degenerate*, if γ maps into $\mathrm{P}_\infty(\mathcal{A})$. The quotient of $\pi_1(\mathrm{AP}_\infty(\mathcal{A}), \mathrm{P}_\infty(\mathcal{A}))$ by the submonoid of homotopy classes of degenerate paths is a group, which will be denoted $\widetilde{\pi}_1(\mathrm{AP}_\infty(\mathcal{A}), \mathrm{P}_\infty(\mathcal{A}))$.

By means of a secondary transgression formula analogous to Eq. (3.1), (cf. [12, Lemma 1.11]), one shows that the even (Chern) character descends to

$$\operatorname{ch}_\bullet : \widetilde{\pi}_1(\mathrm{AP}_\infty(\mathcal{A}), \mathrm{P}_\infty(\mathcal{A})) \to HC_{\mathrm{ev}}(\mathcal{A}, \mathcal{B})\,. \tag{3.3}$$

On the other hand, the obvious homomorphism

$$\widetilde{\pi}_1(\mathrm{AP}_\infty(\mathcal{A}), \mathrm{P}_\infty(\mathcal{A})) \longrightarrow K_0(\mathcal{A}, \mathcal{B}), \quad \gamma \mapsto (\gamma(0), \gamma(1), \sigma \circ \gamma), \tag{3.4}$$

is easily seen to be an isomorphism.

Theorem 3.2 (cf. [8, Sec. 1.6])**.** *Via the canonical isomorphism* (3.4) *and excision in K-theory, the (periodized) character map*

$$\operatorname{ch}_\bullet : K_0(\mathcal{A}, \mathcal{B}) \to HP_{\mathrm{ev}}(\mathcal{A}, \mathcal{B})$$

coincides with the standard Chern character in cyclic homology.

In conjunction with Eqs. (2.8) and (2.13), one finally obtains the desired homotopy invariance of the divisor flow.

Corollary 3.3. *The divisor flow associated to a relative cycle C of degree p over a pair $(\mathcal{A}, \mathcal{B})$, $\mathcal{B} = \mathcal{A}/\mathcal{J}$, of local Banach algebras, can be regarded* *as the pairing of the character of C with the Chern character in relative K-theory. In particular, it defines an additive map*

$$\mathrm{DF}_C : K_i(\mathcal{A}, \mathcal{B}) \to \mathbb{C}, \quad i \equiv p \,(\mathrm{mod}\, 2), \; i = 0, 1.$$

This, of course, applies to the particular situation of parametric pseudodifferential operators. In that case, however, one can prove even more. First of all, since the product $A_s B_s$ of two smooth admissible elliptic paths of (matrices of) parametric pseudodifferential operators is homotopic to $B_s A_s$, the above results together with some additional analytic arguments give the following 'log-additivity' result.

Theorem 3.4 (cf. [8, Thm. 2.6])**.** *Let $A_s \in \mathrm{CL}^m(M, E; \mathbb{R}^{2k+1})$ and $B_s \in \mathrm{CL}^n(M, E; \mathbb{R}^{2k+1})$ with $s \in [0,1]$ and $m, n \in \mathbb{Z}$ be admissible paths of elliptic elements. Then one has the additivity relation*

$$\mathrm{DF}\left((A_s B_s)_{0 \le s \le 1}\right) = \mathrm{DF}\left((A_s)_{0 \le s \le 1}\right) + \mathrm{DF}\left((B_s)_{0 \le s \le 1}\right). \tag{3.5}$$

Secondly, because the algebra of smoothing operators $\mathrm{CL}^{-\infty}(M, E; \mathbb{R}^p)$ has the same K-theory as $\mathbb{C}$, the divisor flows assume integral values. More precisely, in view of the preceding results, and using compatibility of the Bott suspension isomorphisms in K-theory and cyclic cohomology as proved by Elliott–Natsume–Nest [3], the divisor flow pairings acquire the following remarkable topological interpretation.

Theorem 3.5 (cf. [8, Thm. 2.12])**.** *The divisor flow pairing with the character C^p_{reg} of the relative cycle $(\Omega_p, \partial\Omega_p, r, \varrho, \overline{\mathrm{TR}}_p, \widetilde{\mathrm{TR}}_p)$, $p = 2k + i > 0$, implements the Bott isomorphism at the relative K-theory level,*

$$K_i(\mathrm{CL}^0(M, E; \mathbb{R}^{2k+i}), \, \mathrm{C}\mathcal{S}^0(M, E; \mathbb{R}^{2k+i})) \xrightarrow{\simeq} \mathbb{Z}\,, \qquad i = 0, 1\,,$$

in a manner compatible with the Bott suspension.

4. The spectral flow and the divisor flow

In this section we describe the relationship between the spectral flow and the divisor flows, which goes beyond the mere analogy between their properties.

In order to treat both the even and the odd case simultaneously, we make the following notational conventions. Let $p = 2k$ or $p = 2k + 1$ and denote by $c : \mathbb{C}\ell_p \to \mathfrak{M}_{2^k}(\mathbb{C})$ the standard representation of the Clifford algebra $\mathbb{C}\ell_p$. If p is odd then c is irreducible and the image of the volume element is characterized by

$$c(i^{k+1}\, e_1 \, \cdots \, e_{2k+1}) \; = \; \mathrm{Id} =: \gamma,$$

while if p is even, we have

$$c(i^k\, e_1 \, \cdots \, e_{2k}) \; =: \; \gamma\,, \qquad \text{with} \quad \gamma^2 = \mathrm{Id} \quad \text{and} \quad \gamma^* = \gamma\,.$$

So in the even case γ is a (non-trivial) grading operator while in the odd case γ is just the identity.

We now fix an invertible first order self-adjoint elliptic *differential* operator D and define its *p-fold suspension* as

$$\mathcal{D}_p(\mu) := \gamma\left(D \otimes I_{\mathbb{C}^{2^k}} + c(\mu)\right).$$

By construction, $\mathcal{D}_p$ is an element of $\mathrm{CL}^1(M, E \otimes \mathbb{C}^{2^k}; \mathbb{R}^p)$. Since $\mathcal{D}_p(\mu)^2 = D^2 + |\mu|^2$ it follows from the invertibility of D that $\mathcal{D}_p$ is invertible, too.

Recall that the odd parametric η_{2k+1}-invariant was defined by applying (up to a factor) the degree $(2k+1)$-term of the Chern character to the invertible A, cf. Eqs. (1.11), (2.7).

To define the parametric η-invariant in the even case $p = 2k$ we first observe that

$$\mathcal{P} := \frac{1}{2}\Big(I - \big((D^2 + |\operatorname{Id}_{\mathbb{R}^{2k}}|^2)^{-1/2} \otimes I_{\mathbb{C}^{2^k}}\big)\mathcal{D}_{2k}\Big) \tag{4.1}$$

is an idempotent in $\mathrm{CL}^0(M, E \otimes \mathbb{C}^{2^k}; \mathbb{R}^{2k})$. Hence to any invertible first order self-adjoint elliptic differential operator we can naturally associate an idempotent in $\mathrm{CL}^0(M, E \otimes \mathbb{C}^{2^k}; \mathbb{R}^{2k})$. Then, analogously to the odd case the *higher even η-invariant* η_{2k} is defined by applying the degree $2k$-term of the even Chern character to the idempotent $\mathcal{P}$, cf. Eq. (2.11):

$$\eta_{2k}(\mathcal{D}_{2k}) := -\frac{2}{(2\pi i)^k k!}\overline{\mathrm{TR}}_{2k}\left(\big(\mathcal{P} - \frac{1}{2}\big)(d\mathcal{P})^{2k}\right). \tag{4.2}$$

With this understood, Proposition 6.6 in [9] can now be generalized to all dimensions.

Proposition 4.1 (cf. [8, Thm. 3.2])**.** *Let D be an invertible first order self-adjoint elliptic differential operator. Let $\mathcal{D}_p(\mu) := \gamma\left(D \otimes I_{\mathbb{C}^{2^k}} + c(\mu)\right), \mu \in \mathbb{R}^p$, be the p-fold suspension. Then the spectral η-invariant of D equals the parametric η-invariant of $\mathcal{D}_p$:*

$$\eta(D) = \eta_p(\mathcal{D}_p)\,, \quad \textit{for all} \quad p \in \mathbb{N}.$$

In turn, this 'suspension property' of the η-invariants plays a key role in establishing the following relation between the divisor flow and the spectral flow.

Theorem 4.2 (cf. [8, Sec. 3])**.** *Let $D_s : \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(E)$ be a smooth family of elliptic first order self-adjoint differential operators on the vector bundle E such that D_0 and D_1 are invertible. Let $\mathcal{D}_{p,s} := \gamma\left(D_s \otimes I_{\mathbb{C}^{2^k}} + c(\mu)\right)$, $\mu \in \mathbb{R}^{2k}$, be the corresponding family of the p-fold suspensions.*

- **(i)** *If $p = 2k+1$ is odd, then the divisor flow of $(\mathcal{D}_s)_{0\le s\le 1}$ equals the spectral flow of $(D_s)_{0\le s\le 1}$.*
- **(ii)** *If $p = 2k$ is even, then let $\mathcal{P}_s \in \mathrm{CL}^1(M, E \otimes \mathbb{C}^{2^k}; \mathbb{R}^{2k})$ be a smooth family of almost idempotents whose endpoints coincide with the idempotents associated to $\mathcal{D}_j, j = 0, 1$ in Eq. (4.1). The even divisor flow of the family of almost idempotents $(\mathcal{P}_s)_{0\le s\le 1}$ coincides with the spectral flow of $(D_s)_{0\le s\le 1}$.*

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