Math 2001 Section 003, Spring 2021, First Midterm Solutions

1: *(15 points)* For each of the following statements, either prove it is true, or give an explicit counterexample to each of the following statements, and explain briefly how it works.

a. (8 pts) A and B are sets, then $(A \Delta B) - B = A$.

   Solution:

   False. Let $A = \{1, 3\}$ and $B = \{1, 2\}$. Then $B - A = \{2\}$ and $A - B = \{3\}$, so $(A \Delta B) = \{2, 3\}$, and $(A \Delta B) - B = \{3\}$, which is not equal to A.

b. (7 pts) If a, b, and c are integers and ac is divisible by 3 and bc is divisible by 3, then ab is divisible by 3.

   Solution: False. Note that if we take $a = 1$ and $b = 1$, and $c = 3$, then $ac = 3$ is divisible by 3, and $bc = 3$ is divisible by 3. But $bc = 1$, which is not divisible by 3.

2: *(20 points)*

You're walking through the Denver International Airport West Economy Parking Lot one evening, and suddenly you recognize your history professor, who is wandering around the lot, looking worried. Your professor says: "I can't remember where I parked my car! But I do remember a little bit of information about my parking spot. I know that all the parking spots in this lot are labeled by two letters, followed by three numbers. The parking spot for my car had exactly one 'L' (but I don't recall if it was in the first or second place), and as for the three numbers of the parking spot, there was a '5' in the place for the third number, and the remaining two numbers were even. How many possible parking places are there where my car could be parked? And after that, can you help me check them all to find my car?"

You want to help her, so you feel obligated to pull out your notebook and compute.

a. (10 pts) Given that your professor's description of the parking lot labels is correct, and that all letters and all numbers are used for the labeling of spots, how many total labeled parking spots are there in the Denver International Airport West Economy Parking Lot?

Solution: We use the multiplication principle, keeping in mind that there are 26 letters and 10 numerals. There are five total places in the labeling of parking spots: the first two for the letters and the final three for the numbers. Repeats are allowed. Therefore the total number of possible labeled spots are:

$$26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \;=\; (26)^2 \cdot (10)^3.$$

b. (10 pts) Given that your professor's memory is correct, how many of the parking spaces from part (a) are there where her car could be parked?

Solution: We set it up as follows. We know what the license plate is if we can answer the following questions:

1. Where is the L? (2 choices)
2. Having placed the L, what is the the other letter? (25 choices since only one L.)
3. What is the first number, which must be even? (5 choices)
4. What is the second number, which must also be even? (5 choices)

Now we can use the multiplication principle to get

$$2 \times (25) \times (5)^2 \;=\; 2 \times (25)^2.$$

3: *(15 points)*

Consider the power set of $\{w, x, y, z\}$, $2^{\{w,x,y,z\}}$.

1. (5 pts) How many elements does $2^{\{w,x,y,z\}}$ have?

   Solution:

   Since $|\{w, x, y, z\}| = 4$, we know $|2^{\{w,x,y,z\}}| = 2^{|\{w,x,y,z\}|} = 2^4 = 16$.

2. (5 pts) Write out two different elements of $2^{\{w,x,y,z\}}$.

   Solution:

   Elements of $2^{\{w,x,y,z\}}$ are exactly the subsets of $\{w, x, y, z\}$, so two different elements of $2^{\{w,x,y,z\}}$ are $\{x\}$ and $\{y\}$, which are different subsets of $\{w, x, y, z\}$. (There are various other answers.)

3. (5 pts) Give an example of a subset of $2^{\{w,x,y,z\}}$ containing exactly three elements.

   Solution:

   Here is one example (there are various others): $\{\emptyset, \{x\}, \{y\}\}$. Note you need to have set brackets within set brackets to get the right answer.

4: *(15 points)* If a and b are odd integers, prove carefully that $a + b^2$ is even.

Solution:

We suppose a is odd and b is odd.

Since a is odd, there is an integer x such that $a = 2x + 1$.

Since b is odd, there is an integer y such that $b = 2y + 1$.

Therefore we can write

$$\begin{aligned} a + b^2 &= (2x+1) + (2y+1)^2 \\ &= 2x + 1 + 4y^2 + 4y + 1 \\ &= 2x + 4y^2 + 4y + 2 \\ &= 2(x + 2y^2 + 2y + 1). \end{aligned}$$

Let $c = x + 2y^2 + 2y + 1$. Then c is an integer, and $a + b^2 = 2c$.

Thus we have shown $a + b^2$ is even.

5: *(15 points)*

Suppose $A = \{1, 2, 3, 4\}$ and R is an $e$quivalence relation on A.

a. (7 pts) Which of the five properties of relations must R have? State these properties by their mathematical names.

   Solution: By definition, R must be reflexive, transitive, and symmetric.

b. (8 pts) If R contains the elements $\{(3,3), (2,4), (1,4)\}$, what other elements *must* it contain? List them all.

   Solution:

   By reflexivity, it must contain $(1,1), (2,2)$, and $(4,4)$.

   By symmetry, it must contain $(4,2)$ and $(4,1)$.

   By transitivity, it must contain $(2,1)$ and $(1,2)$.

   So the smallest it could be is

   $$R = \{(1,1),\ (2,2),\ (3,3),\ (4,4),\ (1,2),\ (1,4),\ (2,1),\ (2,4),\ (4,1),\ (4,2)\}.$$

6: *(20 points)*

Consider the following statement. 'If you don't take off your hat, either I will show everyone here the Gorilla Glue, or someone here will take the scissors out of the drawer."

a. (7 pts) In terms of the statements $P =$ "you take off your hat," $Q(x) =$ "I show person x the Gorilla Glue," and $R(y) =$ "person y takes the scissors out of the drawer", and the set $S =$ "everyone here," write this sentence with no words, just mathematical symbols.

Solution:

$$(\neg P) \longrightarrow \Big((\forall x \in S, Q(x)) \vee (\exists y \in S, R(y))\Big).$$

b. (8 pts) Write the negation of the statement in mathematical symbols. You must simplify by moving the negation to the right through all quantifiers and Boolean operations to get full credit (i.e., it is not enough to just put a negation symbol in front of the entire statement).

Solution:

The negation of "If A then B" is "A and not B." Thus we have

$$\begin{aligned}
\neg\Big((\neg P) \longrightarrow \Big((\forall x \in S, Q(x)) \vee (\exists y \in S, R(y))\Big)\Big) &= (\neg P) \wedge \neg\Big((\forall x \in S, Q(x)) \vee (\exists y \in S, R(y))\Big) \\
&= (\neg P) \wedge \Big(\neg(\forall x \in S, Q(x)) \wedge \neg(\exists y \in S, R(y))\Big) \\
&= (\neg P) \wedge \Big((\exists x \in S, \neg Q(x)) \wedge (\forall y \in S, \neg R(y))\Big).
\end{aligned}$$

c. (5 pts) Rewrite the negation as an English sentence without any mathematical symbols.

Solution:

"You do not take off your hat, and there's someone here that I don't show the Gorilla Glue, and nobody here takes scissors out of the drawer."