Axioms for an Ordered Field

MATH 3001-002
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Outline

Addition Axioms for $\mathbb{F}$

Let $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F} = \mathbb{R}$.

- A1 For every $x, y \in \mathbb{F}$, $x + y \in \mathbb{F}$, and if $x = w$ and $y = z$, $x + y = w + z$. *(Closure under addition).*
- A2 For every $x, y \in \mathbb{F}$, $x + y = y + x$. *(Commutative Axiom).*
- A3 For every $x, y, z \in \mathbb{F}$, $x + (y + z) = (x + y) + z$. *(Associative Axiom).*
- A4 There exists a unique $0 \in \mathbb{F}$ such that $x + 0 = x$ for all $x \in \mathbb{F}$. *(Existence of additive unit).*
- A5 For every $x \in \mathbb{F}$ there exists a unique $(-x) \in \mathbb{F}$ such that $x + (-x) = 0$. *(Existence of additive inverse).*

Multiplication Axioms for $\mathbb{F}$

- M1 For every $x, y \in \mathbb{F}$, $x \cdot y \in \mathbb{F}$, and if $x = w$ and $y = z$, $x \cdot y = w \cdot z$. *(Closure under multiplication).*
- M2 For every $x, y \in \mathbb{F}$, $x \cdot y = y \cdot x$. *(Commutative Axiom).*
- M3 For every $x, y, z \in \mathbb{F}$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$. *(Associative Axiom).*
- M4 There exists a unique $1 \in \mathbb{F}$ such that $x \cdot 1 = x$ for all $x \in \mathbb{F}$. *(Existence of multiplicative unit).*
- M5 For every $x \in \mathbb{F} - \{0\}$, there exists a unique $(1/x) \in \mathbb{F}$ such that $x \cdot (1/x) = 1$. *(Existence of multiplicative inverse).*
- D For every $x, y, z \in \mathbb{F}$, $x \cdot (y + z) = x \cdot y + x \cdot z$. *(Distributive property of multiplication over addition).*

Order Axioms for $\mathbb{F}$

- O1 For every $x, y \in \mathbb{F}$, exactly one of the following holds: either $x = y$, $x < y$, or $y < x$. *(Trichotomy Law of Order).*
- O2 For every $x, y, z \in \mathbb{F}$, if $x < y$ and $y < z$ then $x < z$. *(Transitive Law of Order).*
- O3 For every $x, y, z \in \mathbb{F}$, if $x < y$ then $x + z < y + z$ *(adding a constant to both sides of an inequality does not change the direction of the inequality).*
- O4 For every $x, y, z \in \mathbb{F}$, if $x < y$ and $0 < z$, then $x \cdot z < y \cdot z$ *(multiplying boths sides of an inequality by a positive constant does not change the direction of the inequality).*