University of Colorado
Department of Mathematics

2019/20 Semester 2 | Math 6320 Real Analysis 2 | Assignment 5

Due Wednesday April 22, 2020

1. Let $X = Y = [0,1]$, and let $\mu = \nu =$ Lebesgue measure m on $[0,1]$. Prove that every open set is in the σ-algebra of subsets of $[0,1] \times [0,1]$ which are measurable with respect to the product Lebesgue measure. Deduce that every Borel subset of $[0,1] \times [0,1]$ is measurable with respect to the product measure $m \times m$.

2. Let
$$f(x,y) = \begin{cases} \frac{x^2-y^2}{(x^2+y^2)^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$
Prove that f is not integrable over $[0,1] \times [0,1]$ with respect to the Lebesgue product measure.

3. Let f and g be real-valued Lebegue measurable functions on [0,1], not assumed to be integrable. Let $E = \{(x,y) \in [0,1] \times [0,1] : f(x) = g(y)\}$.
   (a) Prove that E is measurable with respect to the Lebesgue product measure $m \times m$ defined on $[0,1] \times [0,1]$.
   (Hint: consider the function $F(x,y) = f(x) - g(y)$.)
   (b) Suppose in addition that $m \times m(E) = 1$. Prove that there is a real constant c such that $f \equiv g \equiv c$, m a.e. on $[0,1]$.

4. Do problem #5 in Section 20.1 p. 423 and problem #34 in Section 20.2, p. 437 of the Royden-Fitzpatrick textbook.

5. Let f be as in problem #34, p. 437, done above. Prove that the graph of f,
$$G(f) = \{(x,y) \in \mathbb{R}^2 : f(x) = y\},$$
is a measurable subset of $\mathbb{R}^2$ with respect to the product Lebesgue measure $\mu_2 = m \times m$, and that $\mu_2(G(f)) = 0$.