University of Colorado
Department of Mathematics

2019/20 Semester 2 | Math 6320 Real Analysis 2 | Assignment 1

Due Wednesday Jan. 29, 2020

1. Let G be a countable discrete abelian group and let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ given its relative topology inherited from the standard topology on $\mathbb{C}$. Let $\mathrm{Hom}(G, \mathbb{T})$ be the set of group homomorphisms from G into $\mathbb{T}$ viewed as a group under complex multiplication (the constant function that takes every element of G to 1 is such a homomorphism, so $\mathrm{Hom}(G, \mathbb{T})$ is nonempty). Prove that $\mathrm{Hom}(G, \mathbb{T})$ is itself an abelian group under pointwise operations, and that the topology of pointwise convergence on $\widehat{G} := \mathrm{Hom}(G, \mathbb{T})$ makes $\widehat{G}$ into a compact abelian group. [Hint: consider $\Pi_{g \in G}[\mathbb{T}]_g$.] Prove that $\widehat{G}$ is a compact metric space.

2. Let $K : [0,1] \times [0,1] \to \mathbb{C}$ be continuous. For $f \in C([0,1])$, define

$$T(f)(x) = \int_0^1 K(x,y) f(y) dy.$$

Prove that $T(f) \in C([0,1])$, and that if we define $\overline{B}(\mathbf{0}, 1) = \{f \in C([0,1]) : \|f\| \le 1\}$, then $\{T(f) : f \in \overline{B}(\mathbf{0}, 1)\}$ is precompact in $C([0,1])$.

3. (i) Let G be an open subset of the metric space (X, d). Prove that G is densein X if and only if $X \sim G$ is nowhere dense in X.
   (ii) Let $x \in X$ where (X, d) is a metric space. Prove that $\{x\}$ is nowhere dense in X if and only if x is an accumulation point for X.
   (iii) Prove that if (X, d) is a complete metric space, and each $x \in X$ is an accumulation point for X, then X is uncountable. Deduce that $\mathbb{R}$ is uncountable.

4. Do the following problems in the Royden– Fitzpatrick textbook: pp. 210–211, #2, 4, 6, 11; pp. 214 –215 #19, 22, 25.