University of Colorado
Department of Mathematics

2018/2019 Semester 2 MATH 5330 Second Midterm Exam, Takehome

Due Wednesday April 10, 2019, 5 p.m. in MATH 227.

No late papers will be accepted. You will not receive extra credit for doing the 4330 take-home exam.

INSTRUCTIONS: You are to work by yourself. You are allowed to use the textbook, class notes, previous homework assignments, solutions to the previous midterm exam, and any other book you find helpful; please cite your references. If you need any clarification about a problem, you should consult me, and not other students.

1. (a) Prove that when $x \neq 0,\ \pm 2\pi,\ \pm 4\pi,\ \cdots$, the function

$$D_N(x) = \frac{1}{2} + \sum_{n=1}^{N} \cos(nx)$$

can be expressed as

$$D_N(x) \;=\; \frac{\sin\left[\frac{(2N+1)x}{2}\right]}{2\sin(x/2)}.$$

(b) Prove that if $f : [-\pi, \pi] \to \mathbb{R}$ is piecewise continuous,

$$S_N^f(x) = \frac{a_0(f)}{2} + \sum_{n=1}^{N}[a_n(f)\cos nx + b_n(f)\sin nx] = \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)D_N(s-x)ds,$$

where D_N is as defined in part (a).

(c) Prove that

$$\lim_{N\to\infty}\int_0^{\pi}\left[\frac{1}{x} - \frac{1}{2\sin(x/2)}\right]\sin\left[\frac{(2N+1)x}{2}\right]dx \;=\; 0.$$

[Hint: a result on p. 25 of the textbook may be helpful here.]

(d) Suppose that a function f is piecewise continuous on $[0, \pi]$ and that the right-hand derivative $f'_R(0) = \lim_{h\to 0+}\frac{f(h)-f(0)}{h}$ exists. Prove that

$$\lim_{N\to\infty}\int_0^{\pi} f(s)D_N(s)ds \;=\; \frac{\pi}{2}\lim_{t\to 0+} f(t).$$

2. Suppose that f is continuously differentiable on $[0, \pi]$ with $f(0) = f(\pi) = 0$. Prove that

$$\int_0^{\pi}|f(x)|^2dx \;\leq\; \int_0^{\pi}|f'(x)|^2dx.$$

[Hint: extend f to be an odd periodic function on $[-\pi, \pi]$, and then use Parseval's identity.]

3. Give complete solutions to problems 2.7.5 (p. 123), 5.1.7 (p. 265), 5.2.9 (p. 269), 5.4.3 (a) and (b) (p. 278) and 6.2.5 (p. 311) of the Stade textbook. For Exercise 5.4.3, you may use the result of Exercise 5.2.10 from HW # 9.