Midterm Review Sheet

1 The Three Defining Properties of Real Numbers

For all real numbers a, b and c, the following properties hold true.

1. The **commutative property**:

$$a + b = b + a$$
$$ab = ba$$

2. The **associative property**:

$$a + (b + c) = (a + b) + c$$
$$a(bc) = (ab)c$$

3. The **distributive property**:

$$a(b + c) = ab + ac$$

Remark 1.1 *Recall that for all real numbers a we have*

$$\boxed{-a = (-1)a} \tag{1.1}$$

This is important to remember in the case where you have something like $(a - b)$ in the numerator and $(b - a)$ in the denominator of some rational expression:

$$\frac{a - b}{b - a}$$

You can't cancel, since the top and bottom are different. However, note that by the distributive property and equation (1.1) we have

$$a - b = -1(b - a) = -(b - a)$$

so

$$\frac{a - b}{b - a} = \frac{-1\cancel{(b - a)}}{\cancel{b - a}} = -1$$

These properties define all other facts about real numbers, and chief among them are the formulas we give below.

Remember that the distributive property is an equality between two expressions. Going from the left expression to the right expression is called **distributing** a over the sum $b + c$, while going from the right expression to the left expression is called **factoring** a out, and a is called the **common factor** of ab and ac.

2 Important Formulas

Here are the factoring formulas you should know by now: for any real numbers a and b,

$(a+b)^2 = a^2 + 2ab + b^2$	Square of a Sum
$(a-b)^2 = a^2 - 2ab + b^2$	Square of a Difference
$a^2 - b^2 = (a-b)(a+b)$	Difference of Squares
$a^3 - b^3 = (a-b)(a^2 + 2ab + b^2)$	Difference of Cubes
$a^3 + b^3 = (a+b)(a^2 - 2ab + b^2)$	Sum of Cubes

And here are the exponential rules you should know: for any real numbers a and b, and any rational numbers $\frac{p}{q}$ and $\frac{r}{s}$,

$a^{p/q}a^{r/s} = a^{p/q+r/s} = a^{\frac{ps+qr}{qs}}$	Product Rule
$\frac{a^{p/q}}{a^{r/s}} = a^{p/q-r/s} = a^{\frac{ps-qr}{qs}}$	Quotient Rule
$(a^{p/q})^{r/s} = a^{pr/qs}$	Power of a Power Rule
$(ab)^{p/q} = a^{p/q}b^{p/q}$	Power of a Product Rule
$\left(\frac{a}{b}\right)^{p/q} = \frac{a^{p/q}}{b^{p/q}}$	Power of a Quotient Rule
$a^0 = 1$	Zero Exponent
$a^{-p/q} = \frac{1}{a^{p/q}}$	Negative Exponents
$\frac{1}{a^{-p/q}} = a^{p/q}$	Negative Exponents

Remember, there are different notations:

$$\sqrt[q]{a} = a^{1/q}$$
$$\sqrt[q]{a^p} = a^{p/q} = (a^{1/q})^p$$

For example, the power of a product rule in radical notation would be

$$\sqrt[q]{(ab)^p} = \sqrt[q]{a^p}\sqrt[q]{b^p}$$

Finally, the **quadratic formula**: if a, b and c are real numbers, then the quadratic polynomial equation

$$ax^2 + bx + c = 0$$

has (either one or two) solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The **discriminant** is the number under the square root, $b^2 - 4ac$.

1. If $b^2 - 4ac > 0$, there are two real roots, possibly (indeed quite likely) irrational.
2. If $b^2 - 4ac = 0$, there is one real root, namely $-b/2a$, and it is rational if a and b are.

3. If $b^2 - 4ac < 0$, there are two complex roots, so no real roots.

Remark 2.1 *These formulas give some methods for finding roots of polynomials. For example, the polynomial*

$$x^2 - 5$$

can be factored using the difference of squares formula:

$$(x - 5^{1/2})(x + 5^{1/2})$$

since $5 = 5^1 = 5^{2/2} = (5^{1/2})^2$*. As another example, the polynomial*

$$9x^2 + 6x + 1$$

can be factored using the square of a sum formula:

$$(3x + 1)^2$$

But in general, not all polynomials can be factored using those formulas. Thus, we need the sure-shot quadratic to solve a second degree polynomial like this one:

$$\pi x^2 - \sqrt{2}x + \frac{\sqrt{5}}{2}$$

The quadratic formula was derived using a technique called "completing the square". We discuss this now. ■

3 Completing the Square

In **completing the square**, you start with the "incomplete square" $a^2 + 2ab = c$ and use the sum of squares formula to "complete" it: in practice, you need to find b^2 and add it to both sides. To do that, divide $2ab$ by $2a$ and square the result:

$$\begin{aligned} a^2 + 2ab &= c \\ a^2 + 2ab + \left(\frac{2ab}{2a}\right)^2 &= c + \left(\frac{2ab}{2a}\right)^2 \end{aligned}$$

Then, we have by the square of a sum formula,

$$\left(a + \frac{2ab}{2a}\right)^2 = c + \left(\frac{2ab}{2a}\right)^2$$

i.e.

$$(a + b)^2 = c + b^2$$

We already knew all we had to do was add b^2 to both sides here, but the procedure outlined above is in practice how you find what b^2 even is!

Example 3.1 *Solve* $3x^2 + 7x = 0$ *by completing the square.*

Solution: we don't know what b^2 is here. First, we divide through by 3, so that the x^2 term has a coefficient of 1:

$$x^2 + \frac{7}{3}x = 0$$

Then we divide $\frac{7}{3}x$ by $2x$, just like we divided $2ab$ by $2a$, square the result, and add it to both sides. When we divide by $2x$ and square we get $\left(\frac{7}{6}\right)^2$ or $\frac{49}{36}$:

$$x^2 + \frac{7}{3}x + \frac{49}{36} = \frac{49}{36}$$

so

$$\left(x + \frac{7}{6}\right)^2 = \frac{49}{36}$$

Once the square is completed, we can solve for x, and that is the value of completing the square: Take the square root,

$$x + \frac{7}{6} = \pm\frac{7}{6}$$

Subtract $\frac{7}{6}$ from both sides, and you're done:

$$x = -\frac{7}{6} \pm \frac{7}{6} \qquad \text{or} \qquad \boxed{x = 0, -\frac{7}{3}}$$

Remark 3.2 *Notice we could have done this by straightforward factoring:*

$$3x^2 + 7x = 0 \implies x(3x+7) = 0$$

By the zero product property, either $x = 0$ *or* $3x + 7 = 0$, *so* $x = 0, -\frac{7}{3}$.
Or we could have used the quadratic formula:

$$3x^2 + 7x = 0 \implies x = \frac{-7 \pm \sqrt{7^2 - 4 \cdot 3 \cdot 0}}{2 \cdot 3} = \frac{-7 \pm 7}{6} = 0 \;\; or \;\; -\frac{7}{3}$$

The quadratic formula works in all cases, *but factoring will not always get you to the root in any easy way, as the next example shows. When you have radicals in your roots, it isn't easy to guess them ahead of time. You need a reliable method to get these, and that's what completing the square is really useful for.* ■

Example 3.3 *Solve* $3x^2 + 7x = 2$ *by completing the square.*

Solution: This example is just like Example 3.1, except that we have a 2 instead of a 0 on the right hand side. The procedure is the same, however: divide everything through by 3 to get the x^2 term to have a coefficient of 1:

$$x^2 + \frac{7}{3}x = \frac{2}{3}$$

Then look at $\frac{7}{3}x$, divide it by $2x$, which will give $\frac{7}{6}$, square the result, which gives $\frac{49}{36}$, and add it to both sides:

$$x^2 + \frac{7}{3}x + \frac{49}{36} = \frac{2}{3} + \frac{49}{36}$$

Simplify (on the right side, find a common denominator, which is 36, and add the fractions):

$$\left(x + \frac{7}{6}\right)^2 = \frac{24 + 49}{36} = \frac{73}{36}$$

Take the square roots of both sides:

$$x + \frac{7}{6} = \pm\frac{\sqrt{73}}{6}$$

and subtract $\frac{7}{6}$ from both sides:

$$\boxed{x = -\frac{7}{6} \pm \frac{\sqrt{73}}{6}}$$

4 Exponents

Let's do some examples of taking exponents.

Example 4.1 *Simplify* $\sqrt[3]{-125}$.

Solution: $\sqrt[3]{-125} = [(-5)^3]^{1/3} = (-5)^{3/3} = \boxed{-5}$.

Example 4.2 *Simplify* $x\sqrt[4]{x^5}$.

Solution: $x\sqrt[4]{x^5} = x^1 \cdot x^{5/4} = x^1 \cdot x^1 \cdot x^{1/4} = \boxed{x^2\sqrt[4]{x}}$.

Example 4.3 *Simplify* $12\sqrt[3]{\frac{81}{8z^9}}$.

Solution: $12\sqrt[3]{\frac{81}{8z^9}} = 12\frac{(3^4)^{1/3}}{(2^3)^{1/3}(z^9)^{1/3}} = 12\frac{3\sqrt[3]{3}}{2z^3} = \boxed{\frac{18\sqrt[3]{3}}{z^3}}$.

Example 4.4 *Simplify* $\left(-\frac{1}{2}\right)^{-3}$.

Solution: $\left(-\frac{1}{2}\right)^{-3} = \left(-\frac{2}{1}\right)^{3} = (-2)^3(-1)^3 = \boxed{-8}$.

5 Rational Expressions

Example 5.1 *Compute and simplify* $\frac{y+2}{5y^2+11y+2} + \frac{5}{y^2+y-6}$.

Solution: First, we need to factor the denominators, since then we'll know what our common denominator needs to be:

$$\frac{y+2}{5y^2+11y+2} + \frac{5}{y^2+y-6} = \frac{y+2}{(5y+1)(y+2)} + \frac{5}{(y+3)(y-2)}$$

Well, it looks like we have no choice but to multiply all four of the different factors to get our common denominator:

$$\frac{(y+3)(y-2)}{(y+3)(y-2)} \cdot \frac{y+2}{(5y+1)(y+2)} + \frac{5}{(y+3)(y-2)} \cdot \frac{(5y+1)(y+2)}{(5y+1)(y+2)}$$

$$= \frac{(y+3)(y-2)(y+2)+5(5y+1)(y+2)}{(y+3)(y-2)(5y+1)(y+2)} = \frac{\left[(y+3)(y-2)+5(5y+1)\right]\cancel{(y+2)}}{(y+3)(y-2)(5y+1)\cancel{(y+2)}}$$

$$= \frac{y^2+y-6+25y+5}{(y+3)(y-2)(5y+1)} = \boxed{\frac{y^2+26y-1}{(y+3)(y-2)(5y+1)}}$$

6 Inequalities

Yet another important fact about real numbers, which we exploit quite heavily, is that they are **ordered**. That is, given any real numbers a and b, we always have one of the following relations hold:

$$a \leq b$$
$$b \leq a$$

If $a \neq b$, then one of the following holds:

$$a < b$$
$$b > a$$

When we work with equalities, we can add equal things to both sides, and we can multiply both sides of an equality by any equal expression. For example, we can add 2 to both sides of

$$2x - 2 = -4x + 5$$

to get

$$2x = -4x + 7$$

and we can divide both sides by 2 to get

$$x = -2x + \frac{7}{2}$$

These are called the **additive** and **multiplicative** properties of equalities. The same properties hold for *in*equalities. Thus, if

$$2x - 2 < -4x + 5$$

we can add 2 to both sides to get

$$2x < -4x + 7$$

and we can divide both sides by 2 to get

$$x < -2x + \frac{7}{2}$$

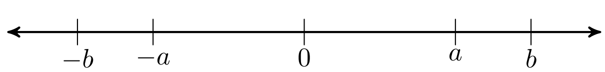
The only difficulty comes when we multiply or divide by a negative number: in that case we have to "flip" the inequality sign. For example, if

$$-5x < 10$$

then

$$x > -2$$

The reason for this can be seen from the following picture: if $a < b$, then $-b < -a$,



Example 6.1 *Solve the inequality* $4(3x - 5) + 18 < 2(5x + 1) + 2x$.

Solution: First, distribute the 4 and the 2 through the parantheses and simplify:

$$\begin{aligned} 4(3x-5)+18 < 2(5x+1)+2x &\iff 12x - 20 + 18 < 10x + 2 + 2x \\ &\iff 12x - 2 < 12x + 2 \end{aligned}$$

Then subtract $12x$ from both sides:

$$\iff \quad -2 < 2$$

This is always true, so the inequality is true for all real x (x doesn't contribute anything here, a multiple of it is just added to both sides, that's all), i.e. the solution is $\mathbb{R}$, otherwise written $\{x \mid x$ is real$\}$ or $(-\infty, \infty)$.

Example 6.2 *Solve and graph the compound inequality* $\frac{3}{5}x + \frac{1}{2} > \frac{3}{10}$ *and* $-4x > 1$.

Solution: Let's do one at a time. First,

$$\begin{aligned} \frac{3}{5}x + \frac{1}{2} > \frac{3}{10} &\iff 6x + 5 > 3 && \text{multiply both sides by the LCD, 10} \\ &\iff 6x > -2 && \text{subtract 5 from both sides} \\ &\iff x > -\frac{1}{3} && \text{divide both sides by 3} \end{aligned}$$

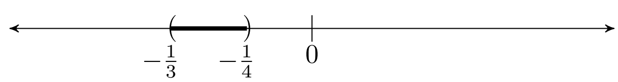
The second inequality is solved by dividing both sides by -4:

$$x < -\frac{1}{4}$$

Thus, the solution set for the compound inequality is

$$\left\{x \mid x > -\frac{1}{3}\right\} \cap \left\{x \mid x < -\frac{1}{4}\right\} = \boxed{\left\{x \mid -\frac{1}{3} < x < -\frac{1}{4}\right\}}$$

Graphically, this is



How do we know $-\frac{1}{3} < -\frac{1}{4}$? Because by the multiplicative property

$$3 < 4 \iff \frac{1}{4} < \frac{1}{3} \iff -\frac{1}{3} < -\frac{1}{4}$$

In interval notation, the solution set is

$$\boxed{\left(-\frac{1}{3}, -\frac{1}{4}\right)}$$

7 Absolute Value

Recall that the absolute value of a real number a is defined as

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$$

The important point to note is that there are *two cases*.

Example 7.1 *Solve the absolute value inequality* $\left|\frac{4x+5}{3} - \frac{1}{2}\right| \leq \frac{7}{6}$.

Solution: Since there are two cases, we have two inequalities to solve (we don't know ahead of time whether the stuff in the absolute value brackets is positive or negative):

$$\frac{4x+5}{3} - \frac{1}{2} \le \frac{7}{6} \tag{7.1}$$

and

$$-\left(\frac{4x+5}{3} - \frac{1}{2}\right) \le \frac{7}{6} \tag{7.2}$$

Starting with (7.1), we have

$$\frac{4x+5}{3} \le \frac{1}{2} + \frac{7}{6} = \frac{10}{6} = \frac{5}{3}$$

so

$$4x + 5 \le 5$$

whence

$$4x \le 0 \qquad \text{or} \qquad x \le 0$$

For the second inequality we have, multiplying both sides by -1,

$$\frac{4x+5}{3} - \frac{1}{2} \ge -\frac{7}{6}$$

so

$$\frac{4x+5}{3} \ge \frac{1}{2} - \frac{7}{6} = -\frac{4}{6} = -\frac{2}{3}$$

Therefore

$$4x + 5 \ge -2$$

so

$$4x \ge -7 \qquad \text{or} \qquad x \ge -\frac{7}{4}$$

Putting these together we get a solution set

$$\boxed{\left\{x \mid -\frac{7}{4} \le x \le 0\right\} \qquad \text{or, in interval notation,} \qquad \left[-\frac{7}{4}, 0\right]}$$

8 Points and Lines

Suppose you have two points in the plane,

$$P = (x_1, y_1), \quad Q = (x_2, y_2)$$

What information can you get from them? Three things:

1. The **distance** between them, $d(P,Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
2. The coordinates of the **midpoint** between them, $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
3. The **slope** of the line through them, $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}}$.

This information comes in handy when doing line problems, especially the slope part. As for **lines**, remember, they can be represented in three different ways:

Standard Form	$ax + by = c$
Slope-Intercept Form	$y = mx + b$
Point-Slope Form	$y - y_1 = m(x - x_1)$

where a, b, c are real numbers, m is the slope, b (different from the standard form b) is the y-intercept, and (x_1, y_1) is *any* fixed point on the line. The first one is basically only useful for finding the x- and y-intercepts (by letting $y = 0$ and $x = 0$, respectively). The second is useful for graphing and finding the x- and y-intercepts (also by letting $y = 0$ and $x = 0$, respectively). The third is useful for coming up with the equation of a line given only information about a point and a slope, or equivalently, by (3), given information about two points.

Suppose two lines ℓ_1 and ℓ_2 have slope-intercept forms $y = m_1 x + b_1$ and $y = m_2 x + b_2$. Then ℓ_1 and ℓ_2 are **parallel**, denoted $\ell_1 \| \ell_2$, if their slopes are the same, that is if $m_1 = m_2$, and they **perpendicular**, denoted $\ell_1 \perp \ell_2$, if their slopes are negative reciprocals, that is if $m_1 = -\frac{1}{m_2}$ (or equivalently $m_2 = -\frac{1}{m_1}$).

Example 8.1 *Suppose I'm given two points*

$$P = (22, 12), \quad Q = (4, -15)$$

and asked to come up with the equation of the line passing through them and then graph it. In order to come up with the equation, I need the slope. I can't do anything without that. Luckily, I can get the slope using (3):

$$m = \frac{-15 - 12}{4 - 22} = \frac{-27}{-18} = \frac{3}{2}$$

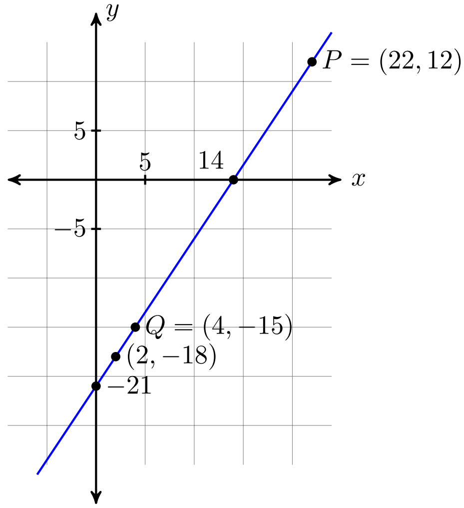
Now I've got a couple of points and I've got a slope, so I naturally use point-slope to give a rough-draft version of the equation (the final draft will be slope-intercept here, since that's what I need to graph it): picking $(22, 12)$ *for no particular reason, I get*

$$y - 12 = \frac{3}{2}(x - 22)$$

Simplifying gives

$$\boxed{y = \frac{3}{2}x - 21}$$

This I can graph: I go to $(0,-21)$, because my y-intercept is -21, which means $x=0$ there, and plot a point there. Then I go up three and over two and plot a point there (i.e. at $(2,-18)$), then I connect the dots, and I'm done.



9 Circles

We know from Euclidean geometry that a **circle**, sometimes denoted $\odot$, is by definition the set of all points $X := (x,y)$ a fixed distance r, called the **radius**, from another given point $C=(h,k)$, called the **center** of the circle,

$$\odot \stackrel{\text{def}}{=} \{X \mid d(X,C)=r\} \tag{9.1}$$

Using the distance formula (1) and the square root property, $d(X,C)=r \iff d(X,C)^2=r^2$, we see that this is precisely

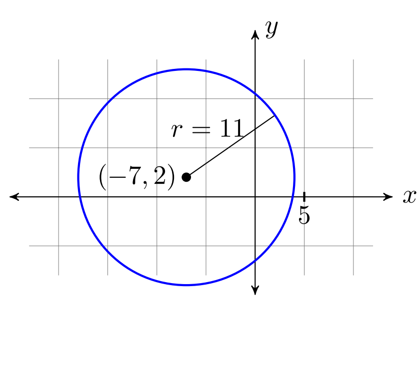
$$\odot \stackrel{\text{def}}{=} \{(x,y) \mid (x-h)^2+(y-k)^2=r^2\} \tag{9.2}$$

which gives the familiar equation for a circle.

Example 9.1 *Suppose $C=(-7,2)$ and $r=11$. The equation of this circle is*

$$\boxed{(x+7)^2+(y-2)^2=121}$$

and it's graph is



10 Functions

An **ordered pair** (a, b) of numbers a and b (or other things, perhaps matrices or polynomials, or whatever) is like the set $\{a, b\}$ except that we're keeping tabs on the positions of a and b. One comes first, the other comes second.

A **cartesian product** $A \times B$ of two sets A and B is the set of all ordered pairs (a, b), i.e.

$$A \times B = \{(a, b) \mid a \in A,\ b \in B\}$$

A **relation** on $A \times B$ is *any* subset of $A \times B$. If $R = A' \times B'$ is a relation on $A \times B$, then the set A' is called the **domain**, and the set B' the **range** of the relation. I.e. the domain is the set of first coordinates, and the range is the set of second coordinates of R. The set B is called the **codomain** of the relation.

Example 10.1 *The set of points a distance greater than* 1 *from* $(0, 0)$ *is a relation on* $\mathbb{R} \times \mathbb{R}$*, since that's a subset of the plane. An ellipse in the plane is a relation on* $\mathbb{R} \times \mathbb{R}$*. Other examples are equality* $=$*, strict inequality* $<$ *and partial inequality* $\leq$ *on* $\mathbb{R} \times \mathbb{R}$*. For example, if we were to graph* $=$*, we'd get the line through the origin with slope of* 1*, i.e.* $y = x$*. Sometimes we have special notation for certain relations. For example, we usually write* $a = b$ *or* $a < b$ *or* $a \leq b$ *instead of* $(a, b) \in =$ *or* $(a, b) \in <$ *or* $(a, b) \in \leq$*, even though* $<$*, for example, really means a subset of the plane, i.e. set* $\{(a, b) \mid a$ *is strictly less than* $b\}$*.*

The most important relation by far is the function. A **function** $f : A \to B$ is actually a relation on $A \times B$, that is, it is a collection of ordered pairs (a, b) in $A \times B$. We employ the special function notation $f(a) = b$ instead of (a, b), to make sure we understand we're dealing with a function here and not just any relation. A function's ordered pairs $f(a) = b$ must satisfy a very important requirement: the "**vertical line test**", which is that if we run a vertical line across the graph of f, we can only hit one point at a time with it. In words, the vertical line test means there are no two points (a, b) and (a, c) in the relation f with $a \neq c$. Remember, an arbitrary relation f is just a bunch of ordered pairs, so it can happen that (a, b) and (a, c) with different b and c are in f. But then we just say f is not a function. Whatever else it is, it's not a function. This is in fact the essence of a function. We *want* functions to avoid having both $f(a) = b$ and $f(a) = c$ for $b \neq c$.

Given a real function f, i.e. a subset of $\mathbb{R} \times \mathbb{R}$, and given two ordered pairs (x_1, y_1) and (x_2, y_2) of f (i.e. $f(x_1) = y_1$ and $f(x_2) = y_2$), we define the **average rate of change** of f as x varies between x_1 and y_1 as the quotient

$$\text{average rate of change} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \tag{10.1}$$

This is a lot like the slope between two points. In fact, it *is* the slope between the two points, except that in this context we pick the points to lie on the graph of f. The reason we call it *average* is we're ignoring all the variation of f between x_1 and x_2, and we're just getting to the bottom line, the net change in y between x_1 and x_2. This is a gross oversimplification of f, but it's much easier to see and compute than *all* of f between those x's.

Sometimes we want to use a **formula** to describe the behavior of a function f. Typically, this is an algebraic expression, but it need not always be so. And it is frequently, though not always, in one variable. In such a case, we usually denote the variable by x, and we write $f(x)$ to denote that the function depends on the variable x, which is appropriately called the **independent variable**. Since to each a in the domain of f we assign one, and only one, b, such that $f(a) = b$, we say that b depends on a. If we have a formula for f, then we assign a variable y to x, satisfying $y = f(x)$, and we say y depends on x, or y is the **dependent variable**.

For example,

$$f(x) = x^2$$

is the parabola. The formula here is x^2 and the function's domain and range are given by the notation $f : \mathbb{R} \to [0, \infty)$. All this is shorthand for the relation explicitly given by the subset of the plane

$$\{(a, a^2) \mid a \in \mathbb{R}\}$$

We often want to graph the function, and this just means plotting the relation in the plane $A \times B$ (usually the x-y plane). An easy way to graph a function g is by transforming the graph of a known function f in such a way as to get g. To do this, we need to know that the graph of a function can be **transformed** in six different ways:

1. **Vertical translation** by k:
$$f(x) \mapsto f(x) \pm k$$
If $+$, the shift is upward, if $-$, it's downward.

2. **Horizontal translation** by h:
$$f(x) \mapsto f(x \pm h)$$
If $+$, the shift is to the left, if $-$, it's to the right.

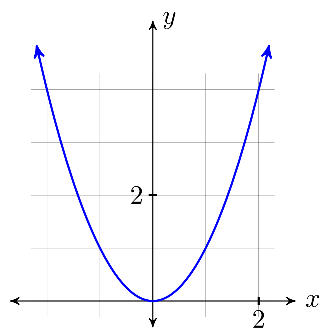
3. **Reflection about the x-axis**:
$$f(x) \mapsto -f(x)$$

4. **Reflection about the y-axis**:
$$f(x) \mapsto f(-x)$$

5. **Vertical stretch or compression**:
$$f(x) \mapsto af(x)$$
where $a > 0$.
   (a) If $0 < a < 1$, this is a vertical compression.
   (b) If $1 < a$, this is a vertical stretch.

6. **Horizontal stretch or compression**:
$$f(x) \mapsto f(ax)$$
where $a > 0$.
   (a) If $0 < a < 1$, this is a horizontal stretch.
   (b) If $1 < a$, this is a horizontal compression.

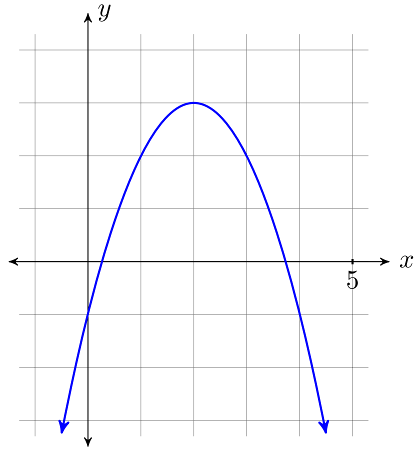
Example 10.2 *For example, the graph of $f(x) = x^2$ is known:*



From this we can get the graph of $g(x) = -x^2 + 4x - 1$. First we complete the square to get

$$g(x) = -(x-2)^2 + 3$$

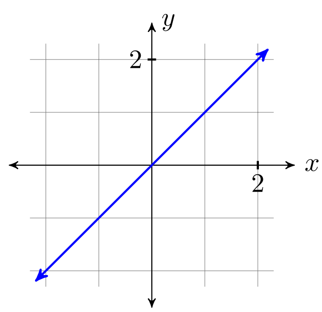
This means that to get the graph of g we take the graph of f, move it right by 2, up by 3, and flip it over:



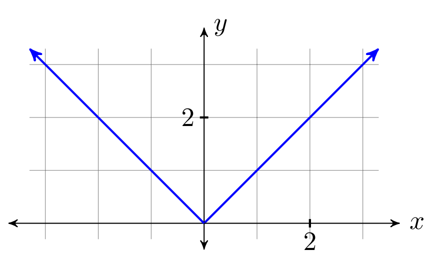
■

Remark 10.3 *This method is only good if you know some basic functions whose graphs you can manipulate. The* **parent functions** *whose graphs you should know are these:*

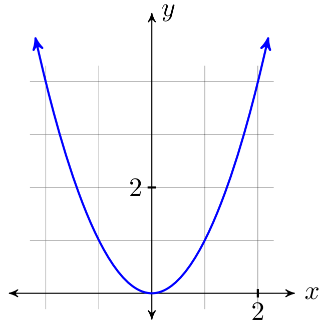
The **diagonal line** *(aka the* **identity function***) $f(x) = x$:*



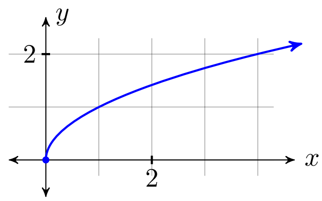
The **absolute value function** *$f(x) = |x|$:*



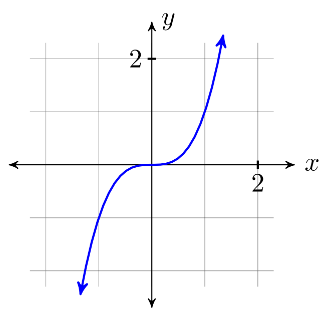
The **parabola** $f(x) = x^2$*:*



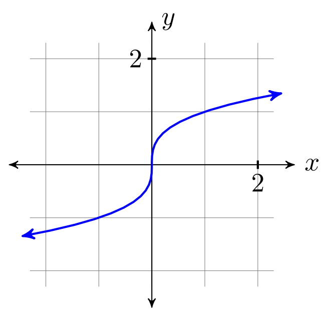
The sideways half parabola, or **square root function**, $f(x) = \sqrt{x}$



The **cube function** $f(x) = x^3$*:*



The sideways cube function, aka the **cube root function**, $f(x) = \sqrt[3]{x}$*:*



■

A function is **even** if it's symmetric about the y-axis, in which case it must satisfy $f(x) = f(-x)$. The parabola $f(x) = x^2$ is an example. A function f is **odd** if it is symmetric about the origin, that is if it satisfies $f(x) = -f(x)$. An example is the cube function $f(x) = x^3$. It is entirely possible that a function is neither even nor odd, for example $f(x) = x^2 + x^3$.

A function may be **increasing**, **decreasing**, or **constant**. It's increasing if

$$x_1 < x_2 \implies f(x_1) < f(x_2)$$

that is if f preserves the order relations of points, or, graphically, if it goes up from left to right. It is decreasing if

$$x_1 < x_2 \implies f(x_1) > f(x_2)$$

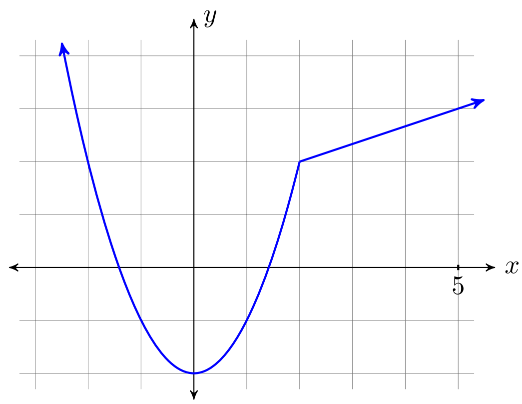
that is if f reverses the order relations of points, or, graphically, if it goes down from left to right. It is constant if it never increases or decreases. Graphically, this would be a horizontal line.

Example 10.4 *The parabola $f(x) = x^2$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$. It is neither increasing nor decreasing at $x = 0$, where it's slope is 0. The cube and cube root functions x^3 and $\sqrt[3]{x}$ are increasing on $(-\infty, 0) \cup (0, \infty)$. The absolute value function $f(x) = |x|$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$. A line $f(x) = mx + b$ with positive slope m is always increasing, and one with negative slope m is always decreasing.* ■

A **piecewise function** is, as the name suggests, a function cobbled together from two or more functions. For example, the function

$$f(x) = \begin{cases} x^2 - 2, & \text{if } \ x \leq 2 \\ \frac{1}{3}x + \frac{4}{3}, & \text{if } \ x > 2 \end{cases}$$

is cobbled together from the parabola $x^2 - 2$ and the line $\frac{1}{3}x + \frac{4}{3}$:



The only question here is the domain of each piece. This is stated in the formulaic definition of the function. For example, in the above example we are told the domain of the parabola is $(-\infty, 2]$ (this is the $x \leq 2$ part), while the domain of the line is $(2, \infty)$ (this is the $x > 2$ part).

We may **add**, **subtract**, **multiply** and **divide** functions $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ (where A and B are subsets of $\mathbb{R}$, and, respectively, the domains of f and g) to get new functions $f + g$, $f - g$, $f \cdot g$ and f/g, and the way we define these new functions is *pointwise*, i.e. point-by-point:

$$\begin{aligned} (f+g)(x) &= f(x) + g(x) \\ (f-g)(x) &= f(x) - g(x) \\ (fg)(x) &= f(x) \cdot g(x) \\ \left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)} \qquad \text{provided } g(x) \neq 0 \end{aligned}$$

Note the domains:

$$\begin{aligned} f+g &: A\cap B \to \mathbb{R} \\ f-g &: A\cap B \to \mathbb{R} \\ fg &: A\cap B \to \mathbb{R} \\ \frac{f}{g} &: A\cap B - \{x \mid g(x)=0\} \to \mathbb{R} \end{aligned}$$

Example 10.5 *Let* $f(x) = \dfrac{x^2-8x+12}{x-6}$ *and let* $g(x)=\sqrt{x-5}$. *Then* $f:(-\infty,6)\cup(6,\infty)\to\mathbb{R}$ *and* $g:[5,\infty)\to\mathbb{R}$, *and*

$$\begin{aligned} f+g &: [5,6)\cup(6,\infty)\to\mathbb{R} \\ f-g &: [5,6)\cup(6,\infty)\to\mathbb{R} \\ fg &: [5,6)\cup(6,\infty)\to\mathbb{R} \\ \frac{f}{g} &: (5,6)\cup(6,\infty)\to\mathbb{R} \end{aligned}$$

have formulas

$$\begin{aligned} (f+g)(x) &= \frac{x^2-8x+12}{x-6}+\sqrt{x-5} \\ (f-g)(x) &= \frac{x^2-8x+12}{x-6}-\sqrt{x-5} \\ (fg)(x) &= \frac{(x^2-8x+12)\sqrt{x-5}}{x-6} \\ \left(\frac{f}{g}\right)(x) &= \frac{x^2-8x+12}{(x-6)\sqrt{x-5}} \end{aligned}$$

■

We may **compose** two functions $f:A\to B$ and $g:B\to C$ to get a new function $g\circ f:A\to C$, and this new function is again defined pointwise:

$$(g\circ f)(x) = g(f(x))$$

The domain of $g\circ f$ is $B\cap f(A)$, that is the intersection of the range of f with the domain of g. We can iterate this process to k different functions, $f_1\circ f_2\circ\cdots\circ f_k$, and we frequently do.

Example 10.6 *Let* $f(x)=x^2-1$ *and* $g(x)=\sqrt{x+3}$. *Then,* $f:\mathbb{R}\to[-1,\infty)$ *and* $g:[-3,\infty)\to[0,\infty)$, *and*

$$\begin{aligned} (f\circ g)(x) &= x+2 \\ (g\circ f)(x) &= \sqrt{x^2+2} \end{aligned}$$

Their domains and ranges are given by

$$\begin{aligned} f\circ g &: [0,\infty)\to[-1,\infty) \\ g\circ f &: [-1,\infty)\to[\sqrt{2},\infty) \end{aligned}$$

■

Functions *may* be, but do not *have* to be, **invertible**. To have an inverse, a function $f:A\to B$ must be **one-to-one**, that is must satisfy the "**horizontal line test**" (in addition to the vertical line test): if $f(a_1)=b$ and $f(a_2)=b$, then $a_1=a_2$, or in other words, unique x-values go with unique y-values. In such a case the **inverse** of f is a function $f^{-1}:B\to A$. An example of a function which is *not* invertible is $f(x)=x^2$. An example of one that is invertibel is $f(x)=x^3$ (it's inverse is $f^{-1}(y)=\sqrt[3]{y}$).

11 Polynomial Functions

11.1 Quadratic Plynomial Functions

These are second degree real polynomial functions whose general formula is

$$f(x) = ax^2 + bx + c \tag{11.1}$$

where a, b and c are real numbers. By completing the square such a function can always be written in the form

$$f(x) = a(x-h)^2 + k \tag{11.2}$$

where $V = (h, k)$ is the coordinate of the **vertex** of the parabola, as we know from our tranformation of the graph of the regular parabola $p(x) = x^2$. In fact, we know more about the vertex. We can get h and k directly from the general equation (11.1):

$$V = (h, k) = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right) \tag{11.3}$$

That is $h = -\frac{b}{2a}$ and $k = f(-\frac{b}{2a})$.

Example 11.1 *The quadratic function $f(x) = x^2 - 4x + 1$ can be written as $f(x) = (x-2)^2 - 3$, so it is a translation of the regular parabola by 2 to the right and 3 down. If we had merely wanted to know where the vertex was, we could have gotten that from equation (11.3):*

$$h = -\frac{b}{2a} = -\frac{-4}{2 \cdot 1} = 2$$

and

$$k = f(2) = 2^2 - 4 \cdot 2 + 1 = -3$$

so $V = (2, -3)$, as we already knew. ■

11.2 Plynomial Functions of Higher Degree

Understanding polynomials of degree greater than 2 becomes a lot more difficult. They're harder to factor and harder to graph. However, there are some techniques.

The first is **long division**. Recall that when we divide two numbers, say $241 \div 5$, we use long division to find the quotient and remainder:

$$\begin{array}{r} 48 \\ 5\,\overline{)\,241} \\ \underline{200} \\ 41 \\ \underline{40} \\ 1 \end{array}$$

In this case 241 is called the **dividend** (that is, the number to be divided), 5 is called the **divisor** (that is, the number doing the dividing), 48 is called the **quotient**, and 1 is called the **remainder**. This information can be summarized by the equation

$$\frac{241}{5} = 48 + \frac{1}{5} \tag{11.4}$$

If we were to multiply both sides of this by the divisor, 5, we would get the alternate form of the equation

$$241 = 5 \cdot 48 + 1 \tag{11.5}$$

In general, the procedure outlined above is called the **division algorithm**: given any two integers p and d, with $d \neq 0$, there exist *unique* integers q and r such that

$$p = dq + r, \qquad \text{or} \qquad \frac{p}{d} = q + \frac{r}{d}, \qquad \text{and where} \qquad 0 \leq r < d \tag{11.6}$$

Here p is the dividend, d the divisor, q the quotient, and r the remainder. Long division is merely a method of obtaining q and r.

The remarkable thing about polynomials is that they, too, obey a **division algorithm**: given any two real polynomials $p(x)$ and $d(x) \neq 0$ (the **dividend** and **divisor**), there exist unique polynomials $q(x)$ and $r(x)$ (the quotient and remainder) such that:

$$p(x) = d(x)q(x) + r(x), \quad \text{or} \quad \frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}, \quad \text{and where} \quad 0 \leq \deg(r(x)) < \deg(d(x)) \tag{11.7}$$

And analogously there is a method of **long division of polynomials** which gives a procedure of obtaining $q(x)$ and $r(x)$. The method is almost identical to that for numbers, except it proceeds by paying attention to the *leading terms* of the dividend and divisor.

Example 11.2 *Let $p(x) = x^3 - 3x^2 - 37$ and $d(x) = x - 5$. Long division will give us $q(x)$ and $r(x)$:*

$$\begin{array}{rrrrr}
 & x^2 & +\,2x & +\,10 & \\ \cline{2-5}
x-5 \big) & x^3 - 3x^2 & & -\,37 & \\
 & -\,x^3 + 5x^2 & & & \\ \cline{2-2}
 & 2x^2 & & & \\
 & -\,2x^2 & +\,10x & & \\ \cline{2-3}
 & & 10x & -\,37 & \\
 & & -\,10x & +\,50 & \\ \cline{3-4}
 & & & 13 &
\end{array}$$

Thus, we know that

$$\underbrace{x^3 - 3x^2 - 37}_{p(x)} = \underbrace{(x-5)}_{d(x)}\underbrace{(x^2 + 2x + 10)}_{q(x)} + \underbrace{13}_{r(x)} \qquad \blacksquare$$

A simpler method, called **synthetic division**, but which only works with degree one divisors, uses only the coefficients of the dividend, and only negative of the constant term in the divisor. The result is the coefficients of the quotient and the remainder.

Example 11.3 *Let us divide the polynomials of the previous example by using synthetic division:*

$$\begin{array}{r|rrrr}
 & 1 & -\,3 & 0 & -\,37 \\
5 & & 5 & 10 & 50 \\ \hline
 & 1 & 2 & 10 & 13
\end{array}$$

The last number, 13, is the remainder, and the other numbers below, 1, 2 and 10, are the coefficients of the quotient $q(x)$. Thus, again,

$$\underbrace{x^3 - 3x^2 - 37}_{1 \;\; -3 \;\; 0 \;\; -37} = \underbrace{(x-5)}_{5}\underbrace{(x^2 + 2x + 10)}_{1 \;\; 2 \;\; 10} + \underbrace{13}_{13} \qquad \blacksquare$$

Some important theorems to note:

Theorem 11.4 (Rational Zeros Theorem) *Let $f(x) = a_n x^2 + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a real polynomial with integer coefficients a_i (that is $a_i \in \mathbb{Z}$). If a rational number p/q is a root, or zero, of $f(x)$, then*

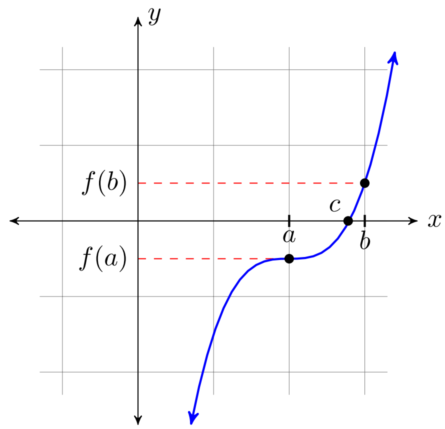
$$p \text{ divides } a_0 \qquad \text{and} \qquad q \text{ divides } a_n$$

■

Theorem 11.5 (Intermediate Value Theorem) *Let $f(x)$ be a real polynomial. If there are real numbers $a < b$ such that $f(a)$ and $f(b)$ have opposite signs, i.e. one of the following holds*

$$f(a) < 0 < f(b)$$
$$f(a) > 0 > f(b)$$

then there is at least one number c, $a < c < b$, such that $f(c) = 0$. That is, $f(x)$ has a root in the interval (a, b).



■

Theorem 11.6 (Remainder Theorem) *If a real polynomial $p(x)$ is divided by $(x - c)$ with the result that*

$$p(x) = (x - c)q(x) + r$$

(r is a number, i.e. a degree 0 polynomial, by the division algorithm mentioned above), then

$$r = p(c)$$

What do we use these theorems for? To narrow the list of guesses as to the possible roots of a given polynomial. The list of guesses is infinite, and we don't want to sit around guessing till the end of time, so having something like the rational zeros theorem and the intermediate value theorem allows us to drastically reduce the number of possibilities. The next example demonstrates how to do this.

Example 11.7 *Find all the zeros of $f(x) = x^5 - 3x^4 + 3x^3 - 5x^2 + 12$.*

Solution: By the Rational Zeros Theorem we know that if there is any rational root p/q at all, then p must divide 12 and q must divide 1. Thus, we don't have to worry about q, since it's only divisors are ± 1. Thus, the possible rational roots are

$$\frac{p}{q} \in \{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12\}$$

We begin with the simplest, 1: by synthetic division we have

$$\begin{array}{r|rrrrrr} & 1 & -3 & 3 & -5 & 0 & 12 \\ 1 & & 1 & -2 & 1 & -4 & -4 \\ \hline & 1 & -2 & 1 & -4 & -4 & 8 \end{array}$$

Since the remainder is 8, we know $(x-1)$ isn't a factor. We move on to $x=-1$:

$$\begin{array}{r|rrrrrr} & 1 & -3 & 3 & -5 & 0 & 12 \\ -1 & & -1 & 4 & -7 & 12 & -12 \\ \hline & 1 & -4 & 7 & -12 & 12 & 0 \end{array}$$

Success! We now know that -1 is a root, and $(x+1)$ is a factor:

$$x^5 - 3x^4 + 3x^3 - 5x^2 + 12 = (x+1)(x^4 - 4x^3 + 7x^2 - 12x + 12)$$

Now to factor the quotient $x^4 - 4x^3 + 7x^2 - 12x + 12$, we repeat the procedure. Luckily $a_4 = 1$ and $a_0 = 12$, as above, so our choices of p/q are the same. We already eliminated 1 as a possibility, so there's no point trying it again. It could be, however, that -1 is a double root, so let's try that:

$$\begin{array}{r|rrrrr} & 1 & -4 & 7 & -12 & 12 \\ -1 & & -1 & 5 & -12 & 24 \\ \hline & 1 & -5 & 12 & -24 & 36 \end{array}$$

Alas, no. Moving on to $x=2$, we find

$$\begin{array}{r|rrrrr} & 1 & -4 & 7 & -12 & 12 \\ 2 & & 2 & -4 & 6 & -12 \\ \hline & 1 & -2 & 3 & -6 & 0 \end{array}$$

Success! Now we know

$$x^5 - 3x^4 + 3x^3 - 5x^2 + 12 = (x+1)(x-2)(x^3 - 2x^2 + 3x - 6)$$

We can factor $x^3 - 2x^2 + 3x - 6$ by grouping:

$$x^3 - 2x^2 + 3x - 6 = (x^3 - 2x^2) + (3x - 6) = x^2(x-2) + 3(x-2) = (x^2+3)(x-2)$$

We have thus completely factored $f(x)$:

$$\begin{aligned} f(x) &= x^5 - 3x^4 + 3x^3 - 5x^2 + 12 \\ &= (x+1)(x-2)(x^2+3)(x-2) \\ &= \boxed{(x+1)(x-2)^2(x^2+3)} \end{aligned}$$

(Note that x^2+3 is irreducible.) ■

Remark 11.8 *It is a fact (which follows from the Fundamental Theorem of Algebra) that every real polynomial $p(x)$ of degree greater than 1 can be factored into a product of linear (that is, degree 1) and irreducible quadratic (that is, degree 2) polynomials,*

$$f(x) = a(x-c_1)^{n_1}(x-c_2)^{n_2}\cdots(x-c_k)^{n_k}(x^2+b_1x+c_1)^{m_1}(x^2+b_2x+c_2)^{m_2}\cdots(x^2+b_\ell x+c_\ell)^{m_\ell}$$

This was illustrated in the above example:

$$x^5 - 3x^4 + 3x^3 - 5x^2 + 12 = (x+1)(x-2)^2(x^2+3)$$ ■

Remark 11.9 *We could have used the remainder theorem on $x = 0$, 1 and -1: $p(0) = 12$, $p(1) = 1-3+3-5+12 = 8$ and $p(-1) = -1-3-3-5-12 = 0$. This would have saved us a little work in finding the first root, $x=-1$.* ■

11.3 Exponential and Logarithmic Functions

A special function we wish to single out is the **exponential function**: given $a > 0$, we define the function $f : \mathbb{R} \to (0, \infty)$, denoted by the formula $f(x) = a^x$, to be a one-to-one, positive, increasing function from all reals to the positive reals, which has the usual exponentiation behavior of numbers for rational x, but which also handles irrational x. For all real x and y the function satisfies the following rules:

$$a^{x+y} = a^x a^y \tag{11.8}$$

$$a^{x-y} = \frac{a^x}{a^y} \tag{11.9}$$

$$a^0 = 1 \tag{11.10}$$

The inverse of the exponential function is the **logarithm**. That is, if $f : \mathbb{R} \to (0, \infty)$ is the exponential $f(x) = a^x$, then its inverse $f^{-1} : (0, \infty) \to \mathbb{R}$ is the logarithm $f(y) = \log_a(y)$. This is the basis for the equivalence of the expressions $y = a^x$ and $\log_a(y) = x$,

$$y = a^x \iff \log_a(y) = x \tag{11.11}$$

In view of the correspondence (11.11), the equations (11.8)-(11.10) for exponentials have analogues for logarithms. These are the **log rules**:

$$\log_a(MN) = \log_a(M) + \log_a(N) \tag{11.12}$$

$$\log_a\left(\frac{M}{N}\right) = \log_a(M) - \log_a(N) \tag{11.13}$$

$$\log_a(1) = 0 \tag{11.14}$$

$$\log_a(M^N) = N \log_a(M) \tag{11.15}$$

The last one, (11.15), follows from the fact that

$$\log_a(M^N) = y \iff a^y = M^N \iff a^{y/N} = M \iff \log_a(M) = y/N \iff N \log_a(M) = y$$

so $\log_a(M^N) = y = N \log_a(M)$.

Logs also satisfy the **change of base formula**: if a, b and c are positive real numbers with $a, b \neq 1$, then

$$\log_b(c) = \frac{\log_a(c)}{\log_a(b)} \tag{11.16}$$

or equivalently

$$\log_a(b) \cdot \log_b(c) = \log_a(c) \tag{11.17}$$

The first tells you that $\log_b(c)$ could be written as a quotient of two logs with a common base a of your choosing, and the second is just another way of writing it (when I was learning this I found the second easy to remember because I thought of it as going from a to b and then from b to c is the same as going from a directly to c. Kind of silly, but it works!).

Example 11.10 *Solve* $3^{2x-1} = 81$.

Solution: $3^{2x-1} = 81 = 3^4$, so since 3^t is a one-to-one function, we know the exponents are equal, $2x - 1 = 4$. From here it's easy: $\boxed{x = \frac{5}{2}}$. ■

Example 11.11 *Solve* $25^{-2x} = 125^{x+7}$.

Solution: We are told $5^{2(-2x)} = 25^{-2x} = 125^{x+7} = 5^{3(x+7)}$, so by the one-to-one property of the exponential, we know $2(-2x) = 3(x+7)$. Simplifying, $-4x = 3x + 21$, or $-21 = 7x$, from which we get $\boxed{x = -3}$. ■

Example 11.12 *Use the properties of logarithms to write* $\log\sqrt{\frac{x}{x+5}}$ *as a sum or difference of simple logarithmic terms.*

Solution:

$$\begin{aligned}
\log\sqrt{\frac{x}{x+5}} &= \log\left(\frac{x}{x+5}\right)^{1/2} \\
&= \frac{1}{2}\log\left(\frac{x}{x+5}\right) \\
&= \boxed{\frac{1}{2}\Big[\log x - \log(x+5)\Big]}
\end{aligned}$$

■

Example 11.13 *Solve* $\log(x+2) = \log 7 + \log x$.

Solution: First, $\log(x+2) = \log 7 + \log x = \log(7x)$. Since the logarithm function is one-to-one, we know $x + 2 = 7x$. Therefore $2 = 6x$, so $\boxed{x = \frac{1}{3}}$. ■