Midterm Review Sheet

1 The Three Defining Properties of Real Numbers

For all real numbers a, b and c, the following properties hold true.

1. The **commutative property**:

$$a + b = b + a$$
$$ab = ba$$

2. The **associative property**:

$$a + (b + c) = (a + b) + c$$
$$a(bc) = (ab)c$$

3. The **distributive property**:

$$a(b + c) = ab + ac$$

Remark 1.1 *Recall that for all real numbers a we have*

$$\boxed{-a = (-1)a} \tag{1.1}$$

This is important to remember in the case where you have something like $(a - b)$ in the numerator and $(b - a)$ in the denominator of some rational expression:

$$\frac{a-b}{b-a}$$

You can't cancel, since the top and bottom are different. However, note that by the distributive property and equation (1.1) we have

$$a - b = -1(b - a) = -(b - a)$$

so

$$\frac{a-b}{b-a} = \frac{-1\cancel{(b-a)}}{\cancel{b-a}} = -1$$

These properties define all other facts about real numbers, and chief among them are the formulas we give below.

Remember that the distributive property is an equality between two expressions. Going from the left expression to the right expression is called **distributing** a over the sum $b + c$, while going from the right expression to the left expression is called **factoring** a out, and a is called the **common factor** of ab and ac.

2 Important Formulas

Here are the factoring formulas you should know by now: for any real numbers a and b,

$(a+b)^2 = a^2 + 2ab + b^2$	Square of a Sum
$(a-b)^2 = a^2 - 2ab + b^2$	Square of a Difference
$a^2 - b^2 = (a-b)(a+b)$	Difference of Squares
$a^3 - b^3 = (a-b)(a^2 + 2ab + b^2)$	Difference of Cubes
$a^3 + b^3 = (a+b)(a^2 - 2ab + b^2)$	Sum of Cubes

And here are the exponential rules you should know: for any real numbers a and b, and any rational numbers $\frac{p}{q}$ and $\frac{r}{s}$,

$a^{p/q}a^{r/s} = a^{p/q+r/s} = a^{\frac{ps+qr}{qs}}$	Product Rule
$\frac{a^{p/q}}{a^{r/s}} = a^{p/q-r/s} = a^{\frac{ps-qr}{qs}}$	Quotient Rule
$(a^{p/q})^{r/s} = a^{pr/qs}$	Power of a Power Rule
$(ab)^{p/q} = a^{p/q}b^{p/q}$	Power of a Product Rule
$\left(\frac{a}{b}\right)^{p/q} = \frac{a^{p/q}}{b^{p/q}}$	Power of a Quotient Rule
$a^0 = 1$	Zero Exponent
$a^{-p/q} = \frac{1}{a^{p/q}}$	Negative Exponents
$\frac{1}{a^{-p/q}} = a^{p/q}$	Negative Exponents

Remember, there are different notations:

$$\boxed{\begin{array}{l} \sqrt[q]{a} = a^{1/q} \\ \sqrt[q]{a^p} = a^{p/q} = (a^{1/q})^p \end{array}}$$

For example, the power of a product rule in radical notation would be

$$\sqrt[q]{(ab)^p} = \sqrt[q]{a^p}\sqrt[q]{b^p}$$

Finally, the **quadratic formula**: if a, b and c are real numbers, then the quadratic polynomial equation

$$ax^2 + bx + c = 0$$

has (either one or two) solutions

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

The **discriminant** is the number under the square root, $b^2 - 4ac$.

1. If $b^2 - 4ac > 0$, there are two real roots, possibly (indeed quite likely) irrational.
2. If $b^2 - 4ac = 0$, there is one real root, namely $-b/2a$, and it is rational if a and b are.

3. If $b^2 - 4ac < 0$, there are two complex roots, so no real roots.

Remark 2.1 *These formulas give some methods for finding roots of polynomials. For example, the polynomial*

$$x^2 - 5$$

can be factored using the difference of squares formula:

$$(x - 5^{1/2})(x + 5^{1/2})$$

since $5 = 5^1 = 5^{2/2} = (5^{1/2})^2$*. As another example, the polynomial*

$$9x^2 + 6x + 1$$

can be factored using the square of a sum formula:

$$(3x + 1)^2$$

But in general, not all polynomials can be factored using those formulas. Thus, we need the sure-shot quadratic to solve a second degree polynomial like this one:

$$\pi x^2 - \sqrt{2}x + \frac{\sqrt{5}}{2}$$

The quadratic formula was derived using a technique called "completing the square". We discuss this now. ■

3 Completing the Square

In **completing the square**, you start with the "incomplete square" $a^2 + 2ab = c$ and use the sum of squares formula to "complete" it: in practice, you need to find b^2 and add it to both sides. To do that, divide $2ab$ by $2a$ and square the result:

$$\begin{aligned} a^2 + 2ab &= c \\ a^2 + 2ab + \left(\frac{2ab}{2a}\right)^2 &= c + \left(\frac{2ab}{2a}\right)^2 \end{aligned}$$

Then, we have by the square of a sum formula,

$$\left(a + \frac{2ab}{2a}\right)^2 = c + \left(\frac{2ab}{2a}\right)^2$$

i.e.

$$(a + b)^2 = c + b^2$$

We already knew all we had to do was add b^2 to both sides here, but the procedure outlined above is in practice how you find what b^2 even is!

Example 3.1 *Solve* $3x^2 + 7x = 0$ *by completing the square.*

Solution: we don't know what b^2 is here. First, we divide through by 3, so that the x^2 term has a coefficient of 1:

$$x^2 + \frac{7}{3}x = 0$$

Then we divide $\frac{7}{3}x$ by $2x$, just like we divided $2ab$ by $2a$, square the result, and add it to both sides. When we divide by $2x$ and square we get $\left(\frac{7}{6}\right)^2$ or $\frac{49}{36}$:

$$x^2 + \frac{7}{3}x + \frac{49}{36} = \frac{49}{36}$$

so

$$\left(x + \frac{7}{6}\right)^2 = \frac{49}{36}$$

Once the square is completed, we can solve for x, and that is the value of completing the square: Take the square root,

$$x + \frac{7}{6} = \pm\frac{7}{6}$$

Subtract $\frac{7}{6}$ from both sides, and you're done:

$$x = -\frac{7}{6} \pm \frac{7}{6} \qquad \text{or} \qquad \boxed{x = 0, -\frac{7}{3}}$$

Remark 3.2 *Notice we could have done this by straightforward factoring:*

$$3x^2 + 7x = 0 \implies x(3x + 7) = 0$$

By the zero product property, either $x = 0$ *or* $3x + 7 = 0$, *so* $x = 0, -\frac{7}{3}$.
Or we could have used the quadratic formula:

$$3x^2 + 7x = 0 \implies x = \frac{-7 \pm \sqrt{7^2 - 4 \cdot 3 \cdot 0}}{2 \cdot 3} = \frac{-7 \pm 7}{6} = 0 \quad or \quad -\frac{7}{3}$$

The quadratic formula works in all cases, *but factoring will not always get you to the root in any easy way, as the next example shows. When you have radicals in your roots, it isn't easy to guess them ahead of time. You need a reliable method to get these, and that's what completing the square is really useful for.* ■

Example 3.3 *Solve* $3x^2 + 7x = 2$ *by completing the square.*

Solution: This example is just like Example 3.1, except that we have a 2 instead of a 0 on the right hand side. The procedure is the same, however: divide everything through by 3 to get the x^2 term to have a coefficient of 1:

$$x^2 + \frac{7}{3}x = \frac{2}{3}$$

Then look at $\frac{7}{3}x$, divide it by $2x$, which will give $\frac{7}{6}$, square the result, which gives $\frac{49}{36}$, and add it to both sides:

$$x^2 + \frac{7}{3}x + \frac{49}{36} = \frac{2}{3} + \frac{49}{36}$$

Simplify (on the right side, find a common denominator, which is 36, and add the fractions):

$$\left(x + \frac{7}{6}\right)^2 = \frac{24 + 49}{36} = \frac{73}{36}$$

Take the square roots of both sides:

$$x + \frac{7}{6} = \pm\frac{\sqrt{73}}{6}$$

and subtract $\frac{7}{6}$ from both sides:

$$\boxed{x = -\frac{7}{6} \pm \frac{\sqrt{73}}{6}}$$

4 Exponents

Let's do some examples of taking exponents.

Example 4.1 *Simplify* $\sqrt[3]{-125}$.

Solution: $\sqrt[3]{-125} = [(-5)^3]^{1/3} = (-5)^{3/3} = \boxed{-5}$.

Example 4.2 *Simplify* $x\sqrt[4]{x^5}$.

Solution: $x\sqrt[4]{x^5} = x^1 \cdot x^{5/4} = x^1 \cdot x^1 \cdot x^{1/4} = \boxed{x^2\sqrt[4]{x}}$.

Example 4.3 *Simplify* $12\sqrt[3]{\frac{81}{8z^9}}$.

Solution: $12\sqrt[3]{\frac{81}{8z^9}} = 12\frac{(3^4)^{1/3}}{(2^3)^{1/3}(z^9)^{1/3}} = 12\frac{3\sqrt[3]{3}}{2z^3} = \boxed{\frac{18\sqrt[3]{3}}{z^3}}$.

Example 4.4 *Simplify* $\left(-\frac{1}{2}\right)^{-3}$.

Solution: $\left(-\frac{1}{2}\right)^{-3} = \left(-\frac{2}{1}\right)^{3} = (-2)^3 = \boxed{-8}$.

5 Rational Expressions

Example 5.1 *Compute and simplify* $\frac{y+2}{5y^2+11y+2} + \frac{5}{y^2+y-6}$.

Solution: First, we need to factor the denominators, since then we'll know what our common denominator needs to be:

$$\frac{y+2}{5y^2+11y+2} + \frac{5}{y^2+y-6} = \frac{y+2}{(5y+1)(y+2)} + \frac{5}{(y+3)(y-2)}$$

Well, it looks like we have no choice but to multiply all four of the different factors to get our common denominator:

$$\frac{(y+3)(y-2)}{(y+3)(y-2)} \cdot \frac{y+2}{(5y+1)(y+2)} + \frac{5}{(y+3)(y-2)} \cdot \frac{(5y+1)(y+2)}{(5y+1)(y+2)}$$
$$= \frac{(y+3)(y-2)(y+2) + 5(5y+1)(y+2)}{(y+3)(y-2)(5y+1)(y+2)} = \frac{\left[(y+3)(y-2) + 5(5y+1)\right]\cancel{(y+2)}}{(y+3)(y-2)(5y+1)\cancel{(y+2)}}$$
$$= \frac{y^2+y-6+25y+5}{(y+3)(y-2)(5y+1)} = \boxed{\frac{y^2+26y-1}{(y+3)(y-2)(5y+1)}}$$

6 Inequalities

Yet another important fact about real numbers, which we exploit quite heavily, is that they are **ordered**. That is, given any real numbers a and b, we always have one of the following relations hold:

$$a \leq b$$
$$b \leq a$$

If $a \neq b$, then one of the following holds:

$$a < b$$
$$b > a$$

When we work with equalities, we can add equal things to both sides, and we can multiply both sides of an equality by any equal expression. For example, we can add 2 to both sides of

$$2x - 2 = -4x + 5$$

to get

$$2x = -4x + 7$$

and we can divide both sides by 2 to get

$$x = -2x + \frac{7}{2}$$

These are called the **additive** and **multiplicative** properties of equalities. The same properties hold for *in*equalities. Thus, if

$$2x - 2 < -4x + 5$$

we can add 2 to both sides to get

$$2x < -4x + 7$$

and we can divide both sides by 2 to get

$$x < -2x + \frac{7}{2}$$

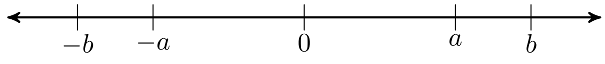
The only difficulty comes when we multiply or divide by a negative number: in that case we have to "flip" the inequality sign. For example, if

$$-5x < 10$$

then

$$x > -2$$

The reason for this can be seen from the following picture: if $a < b$, then $-b < -a$,



Example 6.1 *Solve the inequality* $4(3x - 5) + 18 < 2(5x + 1) + 2x$.

Solution: First, distribute the 4 and the 2 through the parantheses and simplify:

$$\begin{aligned} 4(3x - 5) + 18 < 2(5x + 1) + 2x &\iff 12x - 20 + 18 < 10x + 2 + 2x \\ &\iff 12x - 2 < 12x + 2 \end{aligned}$$

Then subtract $12x$ from both sides:

$$\iff \quad -2 < 2$$

This is always true, so the inequality is true for all real x (x doesn't contribute anything here, a multiple of it is just added to both sides, that's all), i.e. the solution is $\mathbb{R}$, otherwise written $\{x \mid x$ is real$\}$ or $(-\infty, \infty)$.

Example 6.2 *Solve and graph the compound inequality* $\frac{3}{5}x + \frac{1}{2} > \frac{3}{10}$ *and* $-4x > 1$.

Solution: Let's do one at a time. First,

$$\begin{aligned}
\frac{3}{5}x + \frac{1}{2} > \frac{3}{10} \quad &\iff \quad 6x + 5 > 3 && \text{multiply both sides by the LCD, 10} \\
&\iff \quad 6x > -2 && \text{subtract 5 from both sides} \\
&\iff \quad x > -\frac{1}{3} && \text{divide both sides by 3}
\end{aligned}$$

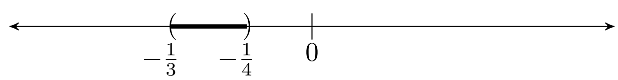
The second inequality is solved by dividing both sides by -4:

$$x < -\frac{1}{4}$$

Thus, the solution set for the compound inequality is

$$\left\{x \mid x > -\frac{1}{3}\right\} \cap \left\{x \mid x < -\frac{1}{4}\right\} = \boxed{\left\{x \mid -\frac{1}{3} < x < -\frac{1}{4}\right\}}$$

Graphically, this is



How do we know $-\frac{1}{3} < -\frac{1}{4}$? Because by the multiplicative property

$$3 < 4 \quad \iff \quad \frac{1}{4} < \frac{1}{3} \quad \iff \quad -\frac{1}{3} < -\frac{1}{4}$$

In interval notation, the solution set is

$$\boxed{\left(-\frac{1}{3}, -\frac{1}{4}\right)}$$

7 Absolute Value

Recall that the absolute value of a real number a is defined as

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$$

The important point to note is that there are *two cases*.

Example 7.1 *Solve the absolute value inequality* $\left|\frac{4x+5}{3} - \frac{1}{2}\right| \leq \frac{7}{6}$.

Solution: Since there are two cases, we have two inequalities to solve (we don't know ahead of time whether the stuff in the absolute value brackets is positive or negative):

$$\frac{4x+5}{3} - \frac{1}{2} \le \frac{7}{6} \tag{7.1}$$

and

$$-\left(\frac{4x+5}{3} - \frac{1}{2}\right) \le \frac{7}{6} \tag{7.2}$$

Starting with (7.1), we have

$$\frac{4x+5}{3} \le \frac{1}{2} + \frac{7}{6} = \frac{10}{6} = \frac{5}{3}$$

so

$$4x + 5 \le 5$$

whence

$$4x \le 0 \qquad \text{or} \qquad x \le 0$$

For the second inequality we have, multiplying both sides by -1,

$$\frac{4x+5}{3} - \frac{1}{2} \ge -\frac{7}{6}$$

so

$$\frac{4x+5}{3} \ge \frac{1}{2} - \frac{7}{6} = -\frac{4}{6} = -\frac{2}{3}$$

Therefore

$$4x + 5 \ge -2$$

so

$$4x \ge -7 \qquad \text{or} \qquad x \ge -\frac{7}{4}$$

Putting these together we get a solution set

$$\boxed{\left\{x \mid -\frac{7}{4} \le x \le 0\right\} \qquad \text{or, in interval notation,} \qquad \left[-\frac{7}{4}, 0\right]}$$