Similarity Classes of Linear Operators

1 Definitions

Two matrices $A, B \in \mathbb{R}^{m\times n}$ are said to be **equivalent matrices** if there exist invertible matrices $P \in \mathrm{GL}(m, \mathbb{R})$ and $Q \in \mathrm{GL}(n, \mathbb{R})$ such that

$$B = PAQ^{-1}$$

Two square matrices $A, B \in \mathbb{R}^{n\times n}$ are said to be **similar matrices** if there exist invertible matrices $P \in \mathrm{GL}(n, \mathbb{R})$ invertible such that

$$B = PAP^{-1}$$

If V is a real vector space and $T, U \in \mathcal{L}(V)$ are linear operators, then T and U are said to be **similar operators** if there is an isomorphism $\phi \in \mathrm{GL}(V)$ such that

$$U = \phi \circ T \circ \phi^{-1}$$

Remark 1.1 *As mentioned in the definitions of* Section 2.1 *of* Abstract Vector Spaces*, similarity of matrices is an* equivalence relation *on* $\mathbb{R}^{n\times n}$*, denoted by* $\sim$*, that is* $A \sim B$ *if* $B = PAQ^{-1}$*, and the equivalence classes,*

$$[A] = \{B \in \mathbb{R}^{2\times 2} \mid A \sim B\}$$

partition the set of all matrices $\mathbb{R}^{2\times 2}$ *into disjoint blocks.*

An identical result holds for the set of linear operators, $\mathcal{L}(V)$*: similarity is an equivalence relation, i.e.* $T \sim U$ *if* T *is similar to* U*, and the equivalence classes*

$$[T] = \{U \in \mathcal{L}(V) \mid T \sim U\}$$

partition $\mathcal{L}(V)$ *into disjoint blocks.*

The question is, what is the relationship between the blocks in $\mathbb{R}^{2\times 2}$ *and those in* $\mathcal{L}(V)$*? We will see that the blocks are in one-to-one correspondence, the correspondence being the result of changing bases.* ■

2 Similarity Classes of Linear Operators

Theorem 2.1 *Let* V *and* W *be finite-dimensional real vector spaces, with dimensions* $\dim(V) = n$ *and* $\dim(W) = m$*. Then any two matrices* $A, B \in \mathbb{R}^{m\times n}$ *are equivalent iff they represent the same linear transformation* $T \in \mathcal{L}(V, W)$*, but possibly with respect to different ordered bases. Moreover, if* A *and* B *are equivalent, then they represent exactly the same set of linear transformations in* $\mathcal{L}(V, W)$*, the* $T \in \mathcal{L}(V, W)$ *varying in accordance with the choice of pairs of ordered bases for* V *and* W*.*

Proof: If A and B represent T, that is if $A = [T]_\beta^\gamma$ and $B = [T]_{\beta'}^{\gamma'}$, for ordered bases β, β' for V and γ, γ' for W, then by the change of bases theorem, Corollary 4.2 in *Abstract Vector Spaces*, we have

$$A = PBQ^{-1}$$

where $P = M_{\gamma,\gamma'}$ and $Q = M_{\beta',\beta}$ are invertible (because they represent an isomorphism on F^n), so that A and B are equivalent. Conversely, if A represents T, that is if $A = [T]_\beta^\gamma$, and A and B are equivalent, then $A = PBQ^{-1}$ for some invertible matrices $P \in \mathbb{R}^{m\times m}$ and $Q \in \mathbb{R}^{n\times n}$. By Theorem 2.10 in *Abstract Vector Spaces*, given P and the ordered basis γ for W, we can get another unique ordered basis γ' for W such that $Q = M_{\gamma,\gamma'}$, and similarly there exists a unique ordered basis β' for W such that $P = M_{\beta,\beta'}$, whence

$$B = QAP^{-1} = M_{\gamma,\gamma'}[T]_\beta^\gamma M_{\beta',\beta} = [T]_{\beta'}^{\gamma'}$$

and B represents T with respect to β' and γ'. Note that it is possible that A represents several transformations, depending on the bases chosen for V and W. By symmetry (i.e. by reversing the roles of A and B in the above argument) we see that A and B represent the same set of linear transformations. ■

Theorem 2.2 *If V is an n-dimensional real vector space, then any two operators $T, U \in \mathcal{L}(V)$ are similar iff there is a matrix $A \in \mathbb{R}^{n\times n}$ that represents both operators, but with respect to possibly different ordered bases. Moreover, if T and U are similar, then they represent exactly the same set of matrices in $\mathbb{R}^{n\times n}$, the $A \in \mathbb{R}^{n\times n}$ varying in accordance with the choice of pairs of ordered bases for V.*

Proof: If T and U are represented by $A \in \mathbb{R}^{n\times n}$, that is if $[T]_\beta = A = [U]_\gamma$ for some ordered bases β and γ for V, then

$$[U]_\gamma = [T]_\beta = M_{\gamma,\beta}[T]_\gamma M_{\beta,\gamma}$$

Defining $\phi : V \to V$ by $\phi(c_i) = b_i$ gives an isomorphism of V with

$$[\phi]_\gamma = \big[[\phi(c_1)]_\gamma \ \cdots \ [\phi(c_n)]_\gamma\big] = \big[[b_1]_\gamma \ \cdots \ [b_n]_\gamma\big] = M_{\beta,\gamma}$$

so that

$$[U]_\gamma = [T]_\beta = M_{\gamma,\beta}[T]_\gamma M_{\beta,\gamma} = [\phi]_\gamma^{-1}[T]_\gamma[\phi]_\gamma = [\phi^{-1}\circ T\circ\phi]_\gamma$$

whence, by uniqueness (Theorem 3.8), $U = \phi\circ T\circ\phi^{-1}$, and $U \sim T$, whence $T \sim U$. Conversely, if $T \sim U$, suppose $A = [T]_\beta$ for some ordered basis β for V. Then using the inverse automorphism $\psi = \phi^{-1}$, we have

$$[U]_\beta = [\psi\circ T\circ\psi^{-1}]_\beta = [\psi]_\beta[T]_\beta[\psi]_\beta^{-1} = M_{\gamma,\beta}[T]_\beta M_{\gamma,\beta}^{-1}$$

whence

$$A = [T]_\beta = M_{\gamma,\beta}^{-1}[U]_\beta M_{\gamma,\beta} = M_{\gamma,\beta}[U]_\beta M_{\gamma,\beta}^{-1} = [U]_\gamma$$

and A represents T and U, but possibly w.r.t. different ordered bases. By symmetry, T and U are represented by the same set of matrices, depending on the ordered bases chosen for V. ■

Remark 2.3 *These two theorems can be summarized as follows: if $\mathcal{S} \subseteq \mathcal{L}(V)$ is a similarity equivalence class, then there is a corresponding class $\mathcal{T} \subseteq M_n(F)$ of all matrices that represent any $T \in \mathcal{S}$, and $\mathcal{S}$ is also the set of all operators in $\mathcal{L}(V)$ that are represented by any $A \in \mathcal{T}$, so that*

$$\mathcal{S} \longleftrightarrow \mathcal{T}$$

which, since $\mathcal{L}(V) \cong M_n(F)$, means $\mathcal{L}(V)$ is partitioned into similarity classes corresponding to exactly the same partitioning in $M_n(F)$. ■