<u>Ex. 3</u>

$$\boxed{\begin{aligned} 2x_1 + 3x_2 + x_3 + 4x_4 - 9x_5 &= 17 \\ x_1 + x_2 + x_3 + x_4 - 3x_5 &= 6 \\ x_1 + x_2 + x_3 + 2x_4 - 5x_5 &= 8 \\ 2x_1 + 2x_2 + 2x_3 + 3x_4 - 8x_5 &= 14 \end{aligned}}$$

or $A\vec{x} = \vec{b}$ where

$$\boxed{A = \begin{pmatrix} 2 & 3 & 1 & 4 & -9 \\ 1 & 1 & 1 & 1 & -3 \\ 1 & 1 & 1 & 2 & -5 \\ 2 & 2 & 2 & 3 & -8 \end{pmatrix} \in M_{4,5}(\mathbb{R}),} \quad \left(\underline{\text{coefficient matrix}}\right)$$

$$\boxed{\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \in \mathbb{R}^5} \quad (\underline{\text{unknown}})$$

$$\boxed{\vec{b} = \begin{pmatrix} 17 \\ 6 \\ 8 \\ 14 \end{pmatrix} \in \mathbb{R}^4} \quad (\underline{\text{constant vector}})$$

Plug it into Wolfram to get

independent variables / linearly dep. columns on the pivot cols.

$$\boxed{B := \operatorname{rref}(A|\vec{b}) = \left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -2 & 3 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)}$$

pivots

From this we see that a basis for the row space is (ignoring the augmented part for now)

$$\boxed{\begin{aligned}\beta_{\substack{\text{row}\\\text{space}\\\text{of }A}} &= (\vec{B}_1, \vec{B}_2, \vec{B}_3) \\ &= (\langle 1,0,2,0,-2\rangle, \langle 0,1,-1,0,1\rangle, \langle 0,0,0,1,-2\rangle)\end{aligned}}$$

Also, columns $\vec{b}_1 = \vec{e}_1$, $\vec{b}_2 = \vec{e}_2$, $\vec{b}_4 = \vec{e}_3 \in \mathbb{R}^4$ are linearly independent, &

$$\left.\begin{aligned}\vec{b}_3 &= 2\vec{b}_1 - \vec{b}_2 \\ b_5 &= -2\vec{b}_1 + \vec{b}_2 - 2\vec{b}_4\end{aligned}\right\} \in \text{span}(\vec{b}_1, \vec{b}_2, \vec{b}_3)$$

so the corresponding columns in A satisfy the same relations,

$$\boxed{\vec{a}_1 = \begin{pmatrix}2\\1\\1\\2\end{pmatrix}, \quad \vec{a}_2 = \begin{pmatrix}3\\1\\1\\2\end{pmatrix}, \quad \vec{a}_4 = \begin{pmatrix}4\\1\\2\\3\end{pmatrix}}$$

are linearly indep., and

$$\boxed{\vec{a}_3 = 2\vec{a}_1 - \vec{a}_2, \quad \vec{a}_5 = -2\vec{a}_1 + \vec{a}_2 - 2\vec{a}_4 \in \text{span}(\vec{a}_1, \vec{a}_2, \vec{a}_4}$$

so a basis for the image of A is

$$\boxed{\beta_{\operatorname{im} A} = (\vec{a}_1, \vec{a}_2, \vec{a}_4) = \left(\begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 2 \\ 3 \end{pmatrix} \right)}$$

We could compute $P = E_p E_{p-1} \cdots E_1 \in GL(4, \mathbb{R})$ by running through the row reduction of $(A | \vec{b})$, & doing this gives

$$\boxed{P = \begin{pmatrix} -1 & 2 & 1 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{pmatrix}}$$

so, according to Wolfram,

$$\boxed{P^{-1} = \begin{pmatrix} 2 & 3 & 4 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 2 & 2 & 3 & 1 \end{pmatrix}}$$

$\vec{p}_1 = \vec{a}_1$, $\vec{p}_2 = \vec{a}_2$, $\vec{p}_3 = \vec{a}_4$, $\vec{p}_4 = \vec{e}_4$

Of course, we could've gotten the 1st 3 columns of P by noting

$$P\vec{b}_1 = P\vec{e}_1 = \vec{p}_1 = \vec{a}_1$$
$$P\vec{b}_2 = P\vec{e}_2 = \vec{p}_2 = a_2$$

but $\vec{p}_4$ we can't get this way.

But as it turns out, we don't need $\vec{P}_4$:

$$\operatorname{im} A \ni \vec{y} = I_4 \vec{y}$$
$$= (P^{-1}P)(A\vec{x})$$
$$= P^{-1}((PA)\vec{x})$$
$$= (\vec{a}_1\ \vec{a}_2\ \vec{a}_4\ \vec{P}_4)(B\vec{x})$$

let's suppose unknown

$$= (\vec{a}_1\ \vec{a}_2\ \vec{a}_4\ \vec{P}_4)\begin{pmatrix} \vec{B}_1 \cdot \vec{x} \\ \vec{B}_2 \cdot \vec{x} \\ \vec{B}_3 \cdot x \\ \vec{B}_4 \cdot \vec{x} \end{pmatrix} \longleftarrow = 0$$

bec. $\vec{B}_4 = \vec{0}$

$$= (\vec{B}_1 \cdot \vec{x})\vec{a}_1 + (\vec{B}_2 \cdot \vec{x})\vec{a}_2 + (\vec{B}_3 \cdot \vec{x})\vec{a}_4 \ \underbrace{(+\ 0\vec{P}_4)}_{=\vec{0}}$$

So again (but slightly differently),

$$\boxed{\begin{aligned} \operatorname{im} A &= \operatorname{span}(\vec{a}_1, \vec{a}_2, \vec{a}_3) \longleftarrow \text{very generally} \\ &= \{(\vec{B}_1 \cdot \vec{x})\vec{a}_1 + (\vec{B}_2 \cdot \vec{x})\vec{a}_2 + (\vec{B}_3 \cdot \vec{x})\vec{a}_4 \mid \vec{x} \in \mathbb{R}^5\} \end{aligned}}$$

* Note that $\vec{B}_1, \vec{B}_2$ & $\vec{B}_3$ span the row space of A, which is orthog. to $\ker A$, so any $\vec{x} \in \ker A$ automatically satisfies

$\vec{B}_i \cdot \vec{x} = 0$. Thus, only $\vec{x} \in \operatorname{span}(\vec{A}_1, \dots, \vec{A}_4)$

$$= \operatorname{span}(\vec{B}_1, \vec{B}_2, \vec{B}_3),$$

i.e. only $\vec{x}$ of the form

$$\boxed{\vec{x} = a\vec{B}_1 + b\vec{B}_2 + c\vec{B}_3, \quad a, b, c \in \mathbb{R}}$$

are sent to a nonzero $\vec{y}$ in im A, & so

$$\text{im } A = \{ (\vec{B}_1 \cdot \vec{x})\vec{a}_1 + (\vec{B}_2 \cdot \vec{x})\vec{a}_2 + (\vec{B}_3 \cdot \vec{x})a_4 \mid \vec{x} \in \mathbb{R}^5 \}$$

$$= \{ a\vec{a}_1 + b\vec{a}_2 + c\vec{a}_4 \mid a, b, c \in \mathbb{R} \}$$
$$= \operatorname{span}(\vec{a}_1, \vec{a}_2, \vec{a}_4)$$

and in fact we gain little by worrying about $\vec{B}_1, \vec{B}_2$ & $\vec{B}_3$ dotting $\vec{x}$! (Except understanding somewhat better why this is the case!)

Finally, the kernel: parametrize $\boxed{x_3 = s, \ x_5 = t} \Rightarrow$

$$\left.\begin{aligned} x_1 &= -2s + 2t \\ x_2 &= s - t \\ x_3 &= 2t \end{aligned}\right\} \Rightarrow \vec{x} = s\underbrace{\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{\vec{u}_1} + t\underbrace{\begin{pmatrix} 2 \\ -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}}_{\vec{u}_2}$$

so a basis for the kernel is

$$\boxed{\beta_{\ker A} = (\vec{u}_1, \vec{u}_2) = \left(\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right)}$$

Since the solutions are merely a coset of ker A (a translation of ker A by a constant vector, $\vec{s}_0 \in \mathbb{R}^4$, any particular solution of $A\vec{x} = \vec{b}$), we merely need to find $\vec{s}_0$ to complete the job. Now, the last column in rref$(A|\vec{b})$ is

$$P\vec{b} = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 0 \end{pmatrix}$$

and since we parametrized x_3 & x_5, leaving x_1, x_2, x_4 dependent on x_3 & x_5, the 3 & the 1 go to the dependent x_1 & x_2, while the 2 goes to x_4, the third dep. variable, giving us

$$\boxed{\vec{s}_0 = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} \in \mathbb{R}^5}$$

Let's verify that in fact $\vec{s}_0 \in S_{A,\vec{b}}$:

$$A\vec{s}_0 = \begin{pmatrix} 2 & 3 & 1 & 4 & -9 \\ 1 & 1 & 1 & 1 & -3 \\ 1 & 1 & 1 & 2 & -5 \\ 2 & 2 & 2 & 3 & -8 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 6+3+8 \\ 3+1+2 \\ 3+1+4 \\ 6+2+6 \end{pmatrix}$$

$$= \begin{pmatrix} 17 \\ 6 \\ 8 \\ 14 \end{pmatrix}$$

$$= \vec{b} \quad \checkmark$$

Thus,

$$\boxed{\begin{aligned} S_{A,\vec{b}} &= \ker A + \vec{s}_0 \qquad \left(= \{\vec{x} + \vec{s}_0 \mid \vec{x} \in \ker A\}\right) \\ &= \operatorname{span}(\vec{u}_1, \vec{u}_2) + \vec{s}_0 \\ &= \left\{ a\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 2 \\ -1 \\ 0 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} \;\middle|\; a, b \in \mathbb{R} \right\} \end{aligned}}$$