Ex. 2

$$\boxed{\begin{aligned} 3x_1 + 3x_2 + 2x_3 + 4x_4 &= 0 \\ 6x_1 + 6x_2 + 2x_3 + 10x_4 &= 0 \end{aligned}}$$

or

$$\boxed{A\vec{x} = \vec{0},} \quad \text{where} \quad \boxed{\begin{aligned} A &= \begin{pmatrix} 3 & 3 & 2 & 4 \\ 6 & 6 & 2 & 10 \end{pmatrix}, \\ \vec{x} &= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4, \\ \vec{0} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathbb{R}^2 \end{aligned}}$$

Then

indp. vars.

$$\boxed{\begin{aligned} B := \operatorname{rref} A &= \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix} \\ &= PA \end{aligned}}$$

pivots

where

$$\boxed{\begin{aligned} P = E_4E_3E_2E_1 &= \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1/2 \end{pmatrix}\begin{pmatrix} 1/3 & 0 \\ 0 & 1 \end{pmatrix}}\underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}} \\ &= \begin{pmatrix} 1/3 & 0 \\ 0 & -1/2 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1/3 & 1/3 \\ 1 & -1/2 \end{pmatrix} \\ &= \frac{1}{6}\begin{pmatrix} -2 & 2 \\ 6 & -3 \end{pmatrix} \end{aligned}}$$

Or we could have computed P^{-1} first, taking the corresponding columns $\vec{a}_1$ & $\vec{a}_3$ of A to those pivot columns $\vec{b}_1$ & $\vec{b}_3$ of B,

$$P^{-1} = \begin{pmatrix} 3 & 2 \\ 6 & 2 \end{pmatrix} = \begin{pmatrix} \vec{a}_1 & \vec{a}_3 \\ | & | \end{pmatrix} \in GL(2, \mathbb{R})$$

and get

$$\boxed{\begin{aligned} P = \frac{1}{\det P^{-1}} \begin{pmatrix} 2 & -2 \\ -6 & 3 \end{pmatrix} &= \frac{-1}{6} \begin{pmatrix} 2 & -2 \\ -6 & 3 \end{pmatrix} \\ &= \frac{1}{6} \begin{pmatrix} -2 & 2 \\ 6 & -3 \end{pmatrix} \end{aligned}}$$

(we used here $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow P^{-1} = \frac{1}{\det P} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$).

Then,

$$\boxed{\beta_{\mathrm{span}(\vec{A}_1, \vec{A}_2)} = \beta_{\substack{\text{row}\\ \text{space}}} = (\vec{B}_1, \vec{B}_2) = (\langle 1,1,0,2 \rangle, \langle 0,0,1,-1 \rangle)}$$

To find $\beta_{\ker A}$ we parametrize x_2 & x_4, the indep. vars.,

$$\begin{aligned} x_2 &= s \\ x_4 &= t \end{aligned} \qquad \begin{aligned} x_1 &= -s-2t \\ x_2 &= t \end{aligned} \qquad \Longrightarrow$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -s+2t \\ s \\ t \\ t \end{pmatrix} = s\underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{=\vec{u}_1} + t\underbrace{\begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}}_{\vec{u}_2}$$

i.e.

$$\boxed{\ker A = S_{A,\vec{0}} = \operatorname{span}\left(\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}\right)}$$

But $\vec{u}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ & $\vec{u}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ are also lin. indep (bec. of slots 2 & 4 (corresp. to the indep. variables x_2 & x_4)),

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \vec{0} = a\vec{u}_1 + b\vec{u}_2 = \begin{pmatrix} -a+2b \\ a \\ b \\ b \end{pmatrix} \Rightarrow a = b = 0$$

so

$$\boxed{\beta_{\ker A} = (\vec{u}_1, \vec{u}_2) = \left(\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}\right)}$$

Finally, since $\vec{b}_1 = \vec{e}_1$, $\vec{b}_3 = \vec{e}_2 \in \mathbb{R}^2$ are the pivot cols., which are linearly dependent, &

$$\vec{b}_2 = \vec{b}_1 \quad \text{and} \quad \vec{b}_4 = 2\vec{b}_1 - \vec{b}_2$$

we have

$$\boxed{\beta_{\operatorname{im} A} = (\vec{a}_1, \vec{a}_3) = \left(\begin{pmatrix} 3 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}\right)}$$

the <u>basis for im A</u>. Moreover,

$$\boxed{\begin{aligned}
\operatorname{im} A \ni \vec{y} &= I_2 \vec{y} \\
&= (P^{-1}P)(A\vec{x}) \\
&= P^{-1}((PA)\vec{x}) \\
&= \begin{pmatrix} | & | \\ \vec{a}_1 & \vec{a}_3 \\ | & | \end{pmatrix}(B\vec{x}) \\
&= \begin{pmatrix} | & | \\ \vec{a}_1 & \vec{a}_3 \\ | & | \end{pmatrix}\begin{pmatrix} \vec{B}_1 \circ \vec{x} \\ \vec{B}_2 \circ \vec{x} \end{pmatrix} \\
&= (\vec{B}_1 \circ \vec{x})\vec{a}_1 + (\vec{B}_2 \circ \vec{x})\vec{a}_2 \\
&= (\langle 1, 1, 0, 2\rangle \circ \vec{x})\vec{a}_1 + (\langle 0, 0, 1, -1\rangle \circ \vec{x})\vec{a}_3 \\
&= (x_1 + x_2 + x_4)\begin{pmatrix} 3 \\ 6 \end{pmatrix} + (x_3 - x_4)\begin{pmatrix} 2 \\ 2 \end{pmatrix}
\end{aligned}}$$

i.e. $\boxed{\operatorname{im} A = \{(\vec{B}_1 \circ \vec{x})\vec{a}_1 + (\vec{B}_2 \circ \vec{x})\vec{a}_3 \mid \vec{x} \in \mathbb{R}^4\}}$ $\boxtimes$

<u>Remark:</u>

$$\boxed{\begin{aligned}
\mathbb{R}^4 &= \underbrace{\ker A}_{\text{2-dim}} \oplus \underbrace{\operatorname{span}(\vec{A}_1, \vec{A}_2)}_{\text{2-dim.}} \\
&= \operatorname{span}(\beta_{\ker A}) \oplus \operatorname{span}(\beta_{\substack{\text{row}\\ \text{space}\\ A}})
\end{aligned}}$$

while $\boxed{\mathbb{R}^2 = \operatorname{im} A = \operatorname{span}(\vec{a}_1, \vec{a}_3).}$ $\boxtimes$