Ex. 1 Let us analyze an example. Consider the homogeneous system

$$\boxed{\begin{aligned} -2x_1 + 3x_2 - 3x_3 - x_4 &= 0 \\ 4x_1 + x_2 + 13x_3 - 5x_4 &= 0 \end{aligned}}$$

which we write

$\boxed{A\vec{x} = \vec{0},}$ where $A = \underbrace{\begin{pmatrix} -2 & 3 & -3 & -1 \\ 4 & 1 & 13 & -5 \end{pmatrix}}_{\text{(coefficient matrix)}} \in M_{2,4}(\mathbb{R})$

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$ (the unknown)

$\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathbb{R}^2$

Let us row reduce A:

$$\begin{pmatrix} -2 & 3 & -3 & -1 \\ 4 & 1 & 13 & -5 \end{pmatrix} \xrightarrow{2I + II} \underbrace{\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}}_{=E_1} \begin{pmatrix} -2 & 3 & -3 & -1 \\ 4 & 1 & 13 & -5 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 3 & -3 & -1 \\ 0 & 7 & 7 & -7 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{7}II} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1/7 \end{pmatrix}}_{=E_2} \begin{pmatrix} -2 & 3 & -3 & -1 \\ 0 & 7 & 7 & -7 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 3 & -3 & -1 \\ 0 & 1 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow{-3\mathrm{II}+\mathrm{I}} \underbrace{\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}}_{=E_3}\begin{pmatrix} -2 & 3 & -3 & -1 \\ 0 & 1 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 & -6 & 2 \\ 0 & 1 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{2}\mathrm{I}} \underbrace{\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}}_{=E_4}\begin{pmatrix} -2 & 0 & -6 & 2 \\ 0 & 1 & 1 & -1 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & -1 \end{pmatrix}}_{=\operatorname{rref} A}$$

Or, in other words

$$\operatorname{rref} A = \underbrace{E_4 E_3 E_2 E_1}_{\equiv P \in GL(2,\mathbb{R})} A = PA$$

column

$$\boxed{\begin{aligned} P = E_4 E_3 E_2 E_1 &= \overbrace{\begin{pmatrix} -1/2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}}^{\text{multiply 1st.}}\begin{pmatrix} 1 & 0 \\ 0 & 1/7 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} -1/2 & 3/2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1/7 \end{pmatrix}}_{\text{mult. next}}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1/2 & 3/14 \\ 0 & 1/7 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1/14 & 3/14 \\ 2/7 & 1/7 \end{pmatrix} \\ &= \frac{1}{14}\begin{pmatrix} -1 & 3 \\ 4 & 2 \end{pmatrix} \end{aligned}}$$

which is easily verified,

$$PA = \frac{1}{14}\begin{pmatrix} -1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} -2 & 3 & -3 & -1 \\ 4 & 1 & 13 & -5 \end{pmatrix}$$

$$= \frac{1}{14}\begin{pmatrix} 14 & 0 & 42 & -14 \\ 0 & 14 & 14 & -14 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & -1 \end{pmatrix}$$

$$= \text{rref}(A)$$

Anyway, what can we say about this? First,

$$\boxed{\begin{aligned} &\dim \text{span}(\vec{A}_1, \vec{A}_2) \\ &\quad = \dim \text{span}(\langle -2, 3, -3, -1\rangle, \langle 4, 1, 13, -5\rangle) \\ &\quad = \dim \text{span}(\langle 1, 0, 3, -1\rangle, \langle 0, 1, 1, -1\rangle) \\ &\quad = \dim \text{span}(\overrightarrow{(\text{rref}A)}_1, \overrightarrow{(\text{rref}A)}_2) \\ &\quad = 2 \\ &\quad = \text{rank}\, A \end{aligned}}$$

and of course, a <u>basis for the row space of A</u> ($\text{span}(\vec{A}_1, \vec{A}_2)$) is $\boxed{\beta_{\text{rows}} = (\langle 1, 0, 3, -1\rangle, \langle 0, 1, 1, -1\rangle)}$.

The row space of A is orthogonal to $\ker A$,

$$\boxed{\operatorname{span}(\vec{A}_1, \vec{A}_2) = (\ker A)^{\perp}}$$

so it remains to describe $\ker A$: parametrize the indep. variables in rref A, which are the x_i's corresponding to the nonpivot columns of rref A,

$$x_3 = s$$
$$x_4 = t$$

then, fr. $\overrightarrow{(\text{rref } A)}_1 = \langle 1, 0, 3, -1 \rangle$ we see that

$$x_1 = -3x_3 + x_4 = -3s + t$$

and fr. $\overrightarrow{(\text{rref } A)}_2 = \langle 0, 1, 1, -1 \rangle$

$$x_2 = -x_3 + x_4 = -s + t$$

Thus,

$$\boxed{\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3s+t \\ -s+t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}}$$

describes the solution space,

$$\boxed{S_{A,\vec{0}} = \ker A = \operatorname{span}\left(\begin{pmatrix}-3\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\1\end{pmatrix}\right)}$$

Moreover, these two vectors, let us call them

$$\boxed{\vec{u} = \begin{pmatrix}-3\\-1\\1\\0\end{pmatrix}, \quad \text{and} \quad \vec{v} = \begin{pmatrix}1\\1\\0\\1\end{pmatrix} \in \mathbb{R}^4}$$

are <u>linearly independent</u>:

If $a\vec{u} + b\vec{v} = \vec{0}$, i.e. if $\begin{pmatrix}-3 & 1\\-1 & 1\\1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$,

then from the 3rd & 4th rows we get

$a = b = 0$, since

$$\begin{pmatrix}0\\0\\0\\0\end{pmatrix} = \begin{pmatrix}-3 & 1\\-1 & 1\\1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}-3a+b\\-a+b\\a\\b\end{pmatrix}.$$

Thus,

$$\boxed{\beta_{\ker A} = (\vec{u}, \vec{v}) = \left(\begin{pmatrix}-3\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\1\end{pmatrix}\right)}$$

is a <u>basis for $\ker A$</u>! (It's linearly indep. and spanning.)

We verify that $\boxed{\text{span}(\vec{A}_1, \vec{A}_2) = (\ker A)^{\perp}}$ by showing that

$$\vec{A}_1 \perp \vec{u}$$
$$\vec{A}_1 \perp \vec{v}$$
$$\vec{A}_2 \perp \vec{u}$$
$$A_2 \perp \vec{v}$$

since then all scalar multiples & other linear combos of $\vec{A}_i$ (or $\vec{u}$ or $\vec{v}$) will also be orthogonal:

$$\vec{A}_1 \cdot \vec{u} = \langle -2, 3, -3, -1 \rangle \cdot \langle -3, -1, 1, 0 \rangle$$
$$= 6 - 3 - 3 - 0 = 0$$

$$\vec{A}_1 \cdot \vec{v} = \langle -2, 3, -3, -1 \rangle \cdot \langle 1, 1, 0, 1 \rangle$$
$$= -2 + 3 - 0 - 1 = 0$$

$$\vec{A}_2 \cdot \vec{u} = \langle 4, 1, 13, -5 \rangle \cdot \langle -3, -1, 1, 0 \rangle$$
$$= -12 - 1 + 13 - 0 = 0$$

$$\vec{A}_2 \cdot \vec{v} = \langle 4, 1, 13, -5 \rangle \cdot \langle 1, 1, 0, 1 \rangle$$
$$= 4 + 1 + 0 - 5 = 0$$

Finally, let us look at the column space of A,

$$\boxed{\begin{aligned} &\operatorname{span}(\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4) \\ &\quad = \operatorname{span}\left(\begin{pmatrix}-2\\4\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}-3\\13\end{pmatrix}, \begin{pmatrix}-1\\-5\end{pmatrix}\right)\end{aligned}}$$

in $\mathbb{R}^2$. Let

$$\boxed{B := \operatorname{rref} A = \begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & -1 \end{pmatrix}}$$

Note, columns

$$\boxed{\begin{aligned} \vec{b}_1 &= \begin{pmatrix}1\\0\end{pmatrix} = \vec{e}_1 \\ \vec{b}_2 &= \begin{pmatrix}0\\1\end{pmatrix} = \vec{e}_2 \end{aligned}}$$

are linearly independent ~~...~~, while columns $\vec{b}_3$ & $\vec{b}_4$ are in the span of $\vec{b}_1$ & $\vec{b}_2$,

$$\boxed{\begin{aligned} \vec{b}_3 &= \begin{pmatrix}3\\1\end{pmatrix} = 3\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} = 3\vec{b}_1 + \vec{b}_2 \\ \vec{b}_4 &= \begin{pmatrix}-1\\-1\end{pmatrix} = -\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} = -\vec{b}_1 - \vec{b}_2 \end{aligned}}$$

But notice that

$$B = PA, \quad \text{where } P = \frac{1}{14}\begin{pmatrix} -1 & 3 \\ 4 & 2 \end{pmatrix} \in GL(2,\mathbb{R})$$

and linear independence is preserved ~~under~~ by an invertible matrix. Indeed,

$$P^{-1} = \begin{pmatrix} -2 & 3 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} | & | \\ \vec{p}_1 & \vec{p}_2 \\ | & | \end{pmatrix}$$

and

$$A = P^{-1}B = P^{-1}(\vec{b}_1\ \vec{b}_2\ \vec{b}_3\ \vec{b}_4)$$

$$= (P^{-1}\vec{b}_1\ \ P^{-1}\vec{b}_2\ \ P^{-1}\vec{b}_3\ \ P^{-1}\vec{b}_4)$$

$$= (\vec{p}_1\ \ \vec{p}_2\ \ P^{-1}(3\vec{b}_1+\vec{b}_2)\ \ P^{-1}(-\vec{b}_1-\vec{b}_2))$$

$$= \Big(\underbrace{\overset{\vec{a}_1}{\vec{p}_1}\ \ \overset{=\vec{a}_2}{\vec{p}_2}}_{\substack{\text{lin}\\ \text{indep.}}}\ \ \underbrace{\overset{=\vec{a}_3}{(3\vec{p}_1+\vec{p}_2)}}_{\substack{\text{in span}\\ \text{of } \vec{p}_1 \,\&\, \vec{p}_2\\ =\vec{a}_1 \,\&\, \vec{a}_2}}\ \ \underbrace{\overset{=\vec{a}_4}{(-\vec{p}_1-\vec{p}_2)}}_{\substack{\text{in span}\\ \text{of } \vec{p}_1 \,\&\, \vec{p}_2 =\\ \vec{a}_1 \,\&\, \vec{a}_2}}\Big)$$

$$= \begin{pmatrix} -2 & 3 & -3 & -1 \\ 4 & 1 & 13 & -5 \end{pmatrix} \checkmark$$

We conclude that

(1) $\underline{\{\vec{a}_1, \vec{a}_2\} = \left\{\begin{pmatrix}-2\\4\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix}\right\}}$ is <u>linearly independent</u> in $\mathbb{R}^2$ (hence forms a basis for $\mathbb{R}^2$)

(2) $\boxed{\vec{a}_3 = 3\vec{a}_1 + \vec{a}_2, \quad \vec{a}_4 = -\vec{a}_1 - \vec{a}_2 \in \operatorname{span}(\vec{a}_1, \vec{a}_2)}$

Hence, since

$$\begin{aligned} \operatorname{im} A &:= \{\vec{y} \in \mathbb{R}^2 \mid \vec{y} = A\vec{x} \text{ for some } \vec{x} \in \mathbb{R}^4\} \\ &= \{A\vec{x} \mid \vec{x} \in \mathbb{R}^4\} \\ &= \{x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 + x_4\vec{a}_4 \mid x_i \in \mathbb{R}\} \\ &= \operatorname{span}(\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4) \\ &= \operatorname{span}(\vec{a}_1, \vec{a}_2) \end{aligned}$$

(since $A\vec{x} = (\vec{a}_1\ \vec{a}_2\ \vec{a}_3\ \vec{a}_4)\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 + x_4\vec{a}_4$)

bec. $\vec{a}_3, \vec{a}_4 \in \operatorname{span}(\vec{a}_1, \vec{a}_2)$ already

we conclude that

$$\boxed{\beta_{\operatorname{im} A} = (\vec{a}_1, \vec{a}_2) = \left(\begin{pmatrix}-2\\4\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix}\right)}$$

is <u>a basis for the image/range of A.</u>

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Remark: Since we have explicit formulas for $\vec{a}_3$ & $\vec{a}_4$,

$$\vec{a}_3 = 3\vec{a}_1 + \vec{a}_2$$
$$\vec{a}_4 = -\vec{a}_1 - a_2$$

we can write any $\vec{y} = A\vec{x} \in \operatorname{im} A$ uniquely in terms of $\vec{a}_1$ & $\vec{a}_2$:

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A\vec{x} = \sum_{j=1}^{4} x_j \vec{a}_j$$

$$= x_1\vec{a}_1 + x_2\vec{a}_2 + x_3(3\vec{a}_1 + \vec{a}_2) + x_4(-\vec{a}_1 - \vec{a}_2)$$

$$= (x_1 + 3x_3 - x_4)\vec{a}_1 + (x_2 + x_3 - x_4)\vec{a}_2$$

Notice that the coefficients $(x_1 + 3x_3 - x_4)$ & $(x_2 + x_3 - x_4)$ can be written as

$$\underbrace{\begin{pmatrix} 1 & 0 & 3 & -1 \end{pmatrix}}_{= \vec{B}_1 = \overrightarrow{\text{rref}(A)}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \;\&\; \begin{pmatrix} 0 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

so that

$$\boxed{\vec{y} = A\vec{x} = (\vec{B}_1 \cdot \vec{x})\vec{a}_1 + (\vec{B}_2 \cdot \vec{x})\vec{a}_2}$$

How should we understand this eqn.? Well, we recall that

- $\vec{y} = A\vec{x}$ (by assump.)
- $P = E_4 E_3 E_2 E_1 \in GL(2,\mathbb{R})$
- $P^{-1} = (\vec{a}_1\ \vec{a}_2) \in GL(2,\mathbb{R})$
- $B = \operatorname{rref} A = PA \iff A = P^{-1}B = \begin{pmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{pmatrix}\begin{pmatrix} - & \vec{B}_1 & - \\ - & \vec{B}_2 & - \end{pmatrix}$

Therefore,

$$\begin{aligned}
\vec{y} = I_2\vec{y} = (P^{-1}P)\vec{y} &= (P^{-1}P)(A\vec{x}) \\
&= \begin{pmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{pmatrix}((PA)\vec{x}) \\
&= \begin{pmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{pmatrix}(B\vec{x}) \\
&= \begin{pmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{pmatrix}\begin{pmatrix} \vec{B}_1 \cdot \vec{x} \\ \vec{B}_2 \cdot \vec{x} \end{pmatrix} \\
&= (\vec{B}_1 \cdot \vec{x})\vec{a}_1 + (\vec{B}_2 \cdot \vec{x})\vec{a}_2
\end{aligned}$$

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i.e.

$$\operatorname{im} A = \operatorname{span}(\vec{a}_1, \vec{a}_2)$$
$$= \{ (\vec{B}_1 \cdot \vec{x})\vec{a}_1 + (\vec{B}_2 \cdot \vec{x})\vec{a}_2 \mid \vec{x} \in \mathbb{R}^4 \}$$

and

$$\boxed{\begin{aligned} \mathbb{R}^4 &= \ker A \oplus \operatorname{span}(\vec{A}_1, \vec{A}_2) \\ &= \operatorname{span}(\beta_{\ker A}) \oplus \operatorname{span}(\beta_{\substack{\text{row}\\ \text{space}\\ \text{of } A}}) \end{aligned}}$$

&

$$\boxed{\begin{aligned} \mathbb{R}^2 = \operatorname{im} A &= \operatorname{span}(\vec{a}_1, \vec{a}_2) \equiv \operatorname{span}(\beta_{\operatorname{im} A}) \\ &= \{ (\vec{B}_1 \cdot \vec{x})\vec{a}_1 + (\vec{B}_2 \cdot \vec{x})\vec{a}_2 \mid \vec{x} \in \mathbb{R}^4 \} \end{aligned}}$$

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