Quiz 4

1. Suppose X is a normal random variable with parameters $\mu = 3$ and $\sigma^2 = 25$. Compute:

   (a) $P(X > 4)$ $= P\left(\frac{X-3}{5} > \frac{4-3}{5}\right) = P(Z > 1/5)$

   $$\boxed{= 1 - \Phi(1/5) \approx 1 - 0.5793 = 0.4207}$$

   (b) $P(1 < X < 10)$ $= P\left(\frac{1-3}{5} < Z < \frac{10-3}{5}\right)$

   $= P(-2/5 < Z < 7/5)$

   $= \Phi(7/5) - \Phi(-2/5)$

   $$\boxed{\begin{aligned} &= \Phi(7/5) - (1 - \Phi(2/5)) \\ &\approx 0.9192 - 1 + 0.6554 = 0.5746 \end{aligned}}$$

2. Suppose X is a normal random variable with mean $\mu = 5$ and $\sigma^2 = 12$. If $P(X > c) = 0.20$, what is the value of c?

   $0.2 = P(X > c) = P\left(Z > \frac{c-5}{\sqrt{12}}\right)$

   $= 1 - \Phi\left(\frac{c-5}{\sqrt{12}}\right)$

   $\Rightarrow \Phi\left(\frac{c-5}{\sqrt{12}}\right) = 0.8$

   $\Rightarrow \frac{c-5}{\sqrt{12}} \approx 0.84$

   $\Rightarrow \boxed{c \approx \sqrt{12}(0.84) + 5 \approx 7.9098}$

3. A continuous random variable X has probability density given by

$$f(x) = \begin{cases} \frac{1}{4}xe^{-x/2}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

(a) Compute $P(X > 5)$.

$$= 1 - P(X \leq 5)$$

$$= 1 - \int_0^5 \frac{1}{4} x e^{-x/2}\,dx \qquad \left(\begin{array}{ll} u = x & dv = e^{-x/2} \\ du = dx & v = -2e^{-x/2} \end{array}\right)$$

$$= 1 - \frac{1}{4}\left[\left[-2xe^{-x/2}\right]_0^5 + 2\int_0^5 e^{-x/2}\,dx\right]$$

$$= 1 - \frac{1}{4}\left[-10e^{-5/2} + 2\left[-2e^{-x/2}\right]_0^5\right]$$

$$= 1 - \frac{1}{4}\left[-10e^{-5/2} \quad \right]$$

$$-4e^{-5/2} + 4 \quad = \boxed{\frac{7}{2}e^{-5/2}}$$

(b) Find $E[X]$.

$$E[X] = \frac{1}{4}\int_0^\infty x^2 e^{-x/2}\,dx \qquad \left(\begin{array}{l} u = x^2 \\ du = 2x\,dx \\ dv = e^{-x/2} \\ v = -2e^{-x/2} \end{array}\right)$$

$$= \frac{1}{4}\left[\left[-2x^2e^{-x}\right]_0^\infty + 4\int_0^\infty xe^{-x/2}\,dx\right]$$

$$\left(\begin{array}{ll} u = x, & du = dx \\ dv = e^{-x/2}, & v = -2e^{-x/2} \end{array}\right)$$

$$= \frac{1}{4}\left[(0-0) + 4\left[-2xe^{-x/2}\right]_0^\infty + 8\int_0^\infty e^{-x/2}\,dx\right]$$

$$= \frac{1}{4}\cdot 8\left[-2e^{-x/2}\right]_0^\infty = 4$$