Quiz 13

MATH 2300-001
November 18, 2008

1. Use the Maclaurin series for $x \sin x$, show that

$$\sum_{k=0}^{\infty} \frac{(-1)^k(2k+2)}{(2k+1)!} = \sin(1) + \cos(1).$$

   - Express $x \sin x$ as its Maclaurin series.
   - Differentiate the function.
   - Differentiate the power series.
   - Relate the new power series with the series we are trying to evaluate.
   - Use this relationship and the derivative of the function to evaluate the series.
   - Fill in the following diagram:

$$\begin{array}{ccc}
x \sin x & = & \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+2} \\
\frac{d}{dx}\big\downarrow & & \big\downarrow \frac{d}{dx} \\
\sin x + x \cos x & = & \sum_{k=0}^{\infty} \frac{(-1)^k(2k+2)}{(2k+1)!} x^{2k+1} \\
x=1\big\downarrow & & \big\downarrow x=1 \\
\sin(1) + \cos(1) & = & \sum_{k=0}^{\infty} \frac{(-1)^k(2k+2)}{(2k+1)!}
\end{array}$$

2. Use a similar method to determine what $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2^k k}$ converges to.

   Hint: Construct a diagram similar to above, starting in the lower right, and fill in any expressions and operations, working counterclockwise.

$$\begin{array}{ccc}
\frac{1}{1-x} & = & \sum_{k=0}^{\infty} x^k = \sum_{k=1}^{\infty} x^{k-1} \\
\int \cdot\, dx\big\downarrow & & \big\downarrow \int \cdot\, dx \\
-\ln(1-x) & = & \sum_{k=1}^{\infty} \frac{1}{k} x^k \\
x=\frac{1}{2}\big\downarrow & & \big\downarrow x=\frac{1}{2} \\
-\ln\left(\frac{1}{2}\right) = \ln 2 & = & \sum_{k=1}^{\infty} \frac{1}{2^k k}
\end{array}$$

3. Solve the differential equation $y'' - y = 0$.

- Suppose a solution to the differential equation can be expressed as a power series:
$$y = \sum_{k=0}^{\infty} a_k x^k$$
- Find power series representations for y' and y''.
- Substitute the formulas into the equation, simplifying if possible.
- Shift the series so that each have the same power of x, and combine.
- Compare coefficients on both sides of the equation to setup recursive relations for a_k. Split into odd and even terms if necessary.
- Express a general term in terms of an early term (say express a_k in terms of a_1).
- Substitute your formulas into the power series for y.
- Identify any common series.

$$y = \sum_{k=0}^{\infty} a_k x^k \qquad y'' = \sum_{k=2}^{\infty} a_k k(k-1)x^{k-2} = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$$

$$0 = y'' - y = \sum_{k=0}^{\infty} (a_{k+2}(k+2)(k+1) - a_k)x^k$$

$$0 = a_{k+2}(k+2)(k+1) - a_k$$

$$a_{k+2} = \frac{a_k}{(k+2)(k+1)}$$

$$a_k = \frac{a_{k-2}}{k(k-1)}$$

$$a_2 = \frac{a_0}{2 \cdot 1} = \frac{a_0}{2!} \qquad a_4 = \frac{a_2}{4 \cdot 3} = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{a_0}{4!} \qquad a_6 = \frac{a_4}{6 \cdot 5} = \frac{a_0}{6!}$$

$$a_3 = \frac{a_1}{3 \cdot 2} = \frac{a_1}{3!} \qquad a_5 = \frac{a_3}{5 \cdot 4} = \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2} = \frac{a_1}{5!} \qquad a_7 = \frac{a_5}{7 \cdot 6} = \frac{a_1}{7!}$$

$$a_{2k} = \frac{a_0}{(2k)!} \qquad a_{2k+1} = \frac{a_1}{(2k+1)!}$$

$$\begin{aligned} y &= \sum_{k=0}^{\infty} a_k x^k \\ &= \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1} \\ &= a_0 \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1} \\ &= a_0 \cosh x + a_1 \sinh x. \end{aligned}$$

4. Repeat with $y'' - y = x^2$.

Same as above except

$$\begin{aligned}
1 &= a_4(4)(3) - a_2 \\
a_4 &= \frac{a_2+1}{(4)(3)} = \frac{\frac{a_0}{2}+1}{3\cdot 4} = \frac{a_0+2}{4!} \\
a_6 &= \frac{a_4}{6\cdot 5} = \frac{a_0+2}{6!} \\
a_{2k} &= \frac{a_0+2}{(2k)!}, \quad k \geq 2 \\
a_2 &= \frac{a_0}{2!} = \frac{a_0+2}{2!} - 1 \\
a_0 &= \frac{a_0+2}{0!} - 2 \\
y &= \sum_{k=0}^{\infty} a_k x^k \\
&= \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1} \\
&= -2 - x^2 + (a_0+2)\sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1} \\
&= -2 - x^2 + (a_0+2)\cosh x + a_1 \sinh x \\
&= c_1 \cosh x + c_2 \sinh x - x^2 - 2.
\end{aligned}$$