Quiz 12

MATH 2300-001
November 11, 2008

1. Determine whether the following series converge conditionally, converge absolutely, or diverge. State and justify all tests you use.

   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{\sin(p)}}$

   $\sin(p) \leq 1$, so the series diverges by p-series.

   (b) $\displaystyle\sum_{k=0}^{\infty} \frac{1}{1000}$

   Diverges by Divergence Test.

   (c) $\displaystyle\sum_{k=1}^{\infty} \frac{k^{\ln k}}{e^k}$

   $$\lim_{k\to\infty} \left(\frac{k^{\ln k}}{e^k}\right)^{\frac{1}{k}} = \lim_{k\to\infty} \frac{k^{\frac{\ln k}{k}}}{e} \overset{\text{L'H}}{=} \frac{1}{e} < 1.$$

   Converges absolutely by Root Test.

   (d) $\displaystyle\sum_{k=2}^{\infty} \ln\left(\frac{k^2}{k^2-1}\right)$

   $$\begin{aligned}\sum_{k=2}^{n} \ln\left(\frac{k^2}{k^2-1}\right) &= \sum_{k=2}^{n} (2\ln k - \ln(k+1) - \ln(k-1)) \\ &= 2\sum_{k=2}^{n} \ln k - \sum_{k=3}^{n+1} \ln k - \sum_{k=1}^{n-1} \ln k \\ &= \ln 2 + \ln n - \ln(n+1) - \ln 1 \\ &= \ln 2 + \ln\left(\frac{n}{n+1}\right).\end{aligned}$$

   $$\sum_{k=2}^{\infty} \ln\left(\frac{k^2}{k^2-1}\right) = \lim_{n\to\infty} \sum_{k=2}^{n} \ln\left(\frac{k^2}{k^2-1}\right) = \ln 2 + \ln 1 = \ln 2.$$

   So, the series converges absolutely (to $\ln 2$).

(e) $\displaystyle\sum_{k=1}^{\infty} \frac{4^{2k}}{k^3(-5)^k}$

$$\lim_{k\to\infty} \left|\frac{4^{2k}}{k^3(-5)^k}\right|^{\frac{1}{k}} = \lim_{k\to\infty} \frac{16}{k^{\frac{3}{k}}5} = \frac{16}{5} > 1.$$

Diverges by Root Test.

(f) $\displaystyle\sum_{k=1}^{\infty} \frac{\cos\left(\ln\left(k^3+(2k)!\right) - \tan^{-1}\left(\sqrt{k+\sin k}\right)\right)}{k^2}$

$$\sum_{k=1}^{\infty} \left|\frac{\cos\left(\ln\left(k^3+(2k)!\right) - \tan^{-1}\left(\sqrt{k+\sin k}\right)\right)}{k^2}\right| \leq \sum_{k=1}^{\infty} \frac{1}{k^2}$$

Converges absolutely by Direct Comparison Test to p-series.

(g) $\displaystyle\sum_{k=1}^{\infty} \ln\left(\frac{k}{3k+1}\right)$

$$\lim_{k\to\infty} \ln\left(\frac{k}{3k+1}\right) = \ln\left(\frac{1}{3}\right) \neq 0$$

Diverges by Divergence Test.

(h) $\displaystyle\sum_{k=1}^{\infty} \frac{k^e}{k^\pi+1}$

$$\sum_{k=1}^{\infty} \frac{k^e}{k^\pi+1} \geq \sum_{k=1}^{\infty} \frac{k^e}{2k^\pi} = \sum_{k=1}^{\infty} \frac{1}{2}\cdot\frac{1}{k^{\pi-e}}$$

Diverges by Direct Comparison Test to p-series.

2. Find the Maclaurin series of π^x.

$f^{(k)}(x) = \pi^x\left(\ln\pi\right)^k.$

$f^{(k)}(0) = \left(\ln\pi\right)^k.$

$\displaystyle\pi^x \approx \sum_{k=0}^{\infty} \frac{\left(\ln\pi\right)^k}{k!}x^k.$