Quiz 11

MATH 2300-001

November 4, 2008

1. Determine whether the following series converge conditionally, converge absolutely, or diverge. State and justify all tests you use.

   (a) $\displaystyle\sum_{k=0}^{\infty} \frac{\sin(k^3-1)}{k^4+1}$

   $$\sum_{k=0}^{\infty} \left| \frac{\sin(k^3-1)}{k^4+1} \right| \leq 1 + \sum_{k=1}^{\infty} \frac{1}{k^4}$$

   Converges Absolutely by Direct Comparison Test to p-series.

   (b) $\displaystyle\sum_{k=1}^{\infty} a_k$ where $a_1 = 1$, $a_{k+1} = \frac{3k+2}{5k-3}a_k$.

   $$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{3k+2}{5k-3} = \lim_{k\to\infty} \frac{3+\frac{2}{k}}{5-\frac{3}{k}} = \frac{3}{5} < 1.$$

   Converges Absolutely by Ratio Test.

   (c) $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln k}$

   $\ln k$ is monotone increasing, so $\frac{1}{\ln k}$ is monotone decreasing. Also, $\lim_{k\to\infty} \frac{1}{\ln k} = \frac{1}{\infty} = 0$. So, by the Alternating Series Test, $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln k}$ converges.

   Let $x \geq 1$. Then $\frac{1}{x} - 1 \leq 0$. So, $\ln x - x$ is monotone decreasing for $x \geq 1$. At $x = 1$ we have $\ln x - x = 0 - 1 = -1 < 0$. So, $\ln x - x < 0$ for all $x \geq 1$. Then $\ln x < x$, and so $\frac{1}{\ln x} > \frac{1}{x}$ for $x \geq 1$. Thus, $\displaystyle\sum_{k=2}^{\infty} \frac{1}{\ln k} \geq \sum_{k=2}^{\infty} \frac{1}{k}$, a divergent p-series. Therefore, by the Direct Comparison Test, $\displaystyle\sum_{k=2}^{\infty} \frac{1}{\ln k}$ diverges. Hence, $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln k}$ converges conditionally.

2. For each of the following, give an example of a series $\sum a_k$ satisfying the condition(s).

(a) $\lim_{k\to\infty} a_k \neq 0$.

(b) $\lim_{k\to\infty} a_k = 0$ but $\sum a_k$ diverges.

(c) $\sum a_k$ converges, but not absolutely.

(d) $\sum a_k$ converges absolutely.

(a) $\sum_{k=1}^{\infty} k$

(b) $\sum_{k=1}^{\infty} \frac{1}{k}$

(c) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$

(d) $\sum_{k=1}^{\infty} \frac{1}{k^2}$

3. For $p > 1$, evaluate $\dfrac{1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + - \cdots}$.

$$A = \frac{1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + - \cdots} = \frac{\sum \frac{1}{k^p}}{\sum \frac{(-1)^{k+1}}{k^p}} = \frac{\sum \frac{(-1)^{k+1}}{k^p} + 2\sum \frac{1}{(2k)^p}}{\sum \frac{(-1)^{k+1}}{k^p}}$$

$$= 1 + 2\frac{\sum \frac{1}{(2k)^p}}{\sum \frac{(-1)^{k+1}}{k^p}} = 1 + 2\frac{1}{2^p} \cdot \frac{\sum \frac{1}{k^p}}{\sum \frac{(-1)^{k+1}}{k^p}} = 1 + \frac{1}{2^{p-1}} A$$

$$\left(1 - \frac{1}{2^{p-1}}\right) A = 1$$

$$A = \frac{1}{1 - \frac{1}{2^{p-1}}} = \frac{2^{p-1}}{2^{p-1} - 1}.$$