Quiz 10

MATH 2300-001

October 28, 2008

1. $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k^2 + \sin\left(\frac{1}{k}\right)}}{\tan^{-1}(k^5+1)}$

$$\lim_{k\to\infty} \frac{\sqrt{k^2 + \sin\left(\frac{1}{k}\right)}}{\tan^{-1}(k^5+1)} = \frac{\sqrt{\infty + 0}}{\frac{\pi}{2}} = \infty \neq 0.$$

So, by the divergence test, the series diverges.

2. Suppose $\displaystyle\sum_{k=1}^{n} a_k = \frac{n-1}{n+1}$ for all n. Find a_n and $\displaystyle\sum_{k=1}^{\infty} a_k$.

$$a_n = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k = \frac{n-1}{n+1} - \frac{n-2}{n} = \frac{2}{n(n+1)}.$$

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} \frac{n-1}{n+1} = 1.$$

3. Does $\displaystyle\sum_{k=1}^{\infty} \frac{5+\cos(\pi k)}{k^{2/3}+k^{1/3}}$ converge or diverge? Why?

$$\sum_{k=1}^{\infty} \frac{5+\cos(\pi k)}{k^{2/3}+k^{1/3}} \geq \sum_{k=1}^{\infty} \frac{5-1}{k^{2/3}+k^{2/3}} = \sum_{k=1}^{\infty} \frac{2}{k^{2/3}}.$$

This is a divergent p-series, so the original series diverges by the Direct Comparison Test.

4. If $\sum a_k$ is a convergent series with positive terms, does $\sum \sin(a_k)$ converge? Does $\sum \cos(a_k)$? Why?

$\sum a_k$ Converges $\Rightarrow \displaystyle\lim_{k\to\infty} a_k = 0 \Rightarrow \lim_{k\to\infty} \cos(a_k) = \cos(0) = 1 \neq 0$. So, by the Divergence Test, $\sum \cos(a_k)$ diverges.

$\displaystyle\lim_{k\to\infty} \frac{\sin(a_k)}{a_k} = 1$ since $a_k \to 0$. Since $\sum a_k$ converges, by Limit Comparison Test, $\sum \sin(a_k)$ converges.

5. Does $\sum_{k=1}^{\infty} \frac{k!}{e^{k^2}}$ converge or diverge? Why?

$$\lim_{k\to\infty} \frac{(k+1)!}{e^{(k+1)^2}} \cdot \frac{e^{k^2}}{k!} = \lim_{k\to\infty} \frac{k+1}{e^{(k+1)^2-k^2}} = \lim_{k\to\infty} \frac{k+1}{e^{2k+1}} = \lim_{x\to\infty} \frac{x+1}{e^{2x+1}} = \lim_{x\to\infty} \frac{1}{2e^{2x+1}} = 0.$$

So, the series converges by the Ratio Test.