Sequences of Real Numbers, Part II:

Infinite Series – Basic Convergence Tests

Def 1 Infinite series are "infinite sums,"

$$\boxed{\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots}$$

but this must be interpreted as a limit,

$$\boxed{\begin{aligned} \sum_{n=1}^{\infty} a_n &\stackrel{def}{=} \lim_{N\to\infty} \sum_{n=1}^{N} a_n \\ &= \lim_{N\to\infty} s_N \end{aligned}}$$

in fact the limit of the sequence s_n of partial sums,

$$\boxed{s_N \stackrel{def}{=} \sum_{n=1}^{N} a_n}$$

e.g.

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_N &= a_1 + a_2 + \cdots + a_N \end{aligned}$$

As with sequences in general, so with the sequences of partial sums: Do they converge? Because the sequences are of a special type, namely finite sums, we can state several extra convergence tests:

In addition to the squeeze theorem, which incidentally proves the comparison test, and the monotone sequence theorem, we have several new tools:

(1) Divergence Test
(2) Integral Test
(3) p-Series Test
(4) Comparison Test
(5) Limit Comparison Test
(6) Alternating Test
(7) Ratio Test

There are yet other tests, but let us focus on these for now.

Basic Properties Deriving From Generalities about Sequences (see Part I)

(1) $\boxed{s_n \to s \overset{\text{def}}{=} \sum_{n=1}^{\infty} a_n}$ by definition means,

for any error size $\varepsilon > 0$ we choose,

$$\boxed{|s_n - s| < \varepsilon \quad \text{if } n \geq N \text{ for some fixed } N.}$$

But $|s_n - s| = |s - s_n| = \left| \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n \right|$

the full series

Nth part. sum

$$= \left| \sum_{n=N+1}^{\infty} a_n \right|$$

we've subtracted off the 1st N terms

$$= \lim_{K \to \infty} \left| \sum_{n=N+1}^{N+K} a_n \right|$$

so

$$\lim_{K \to \infty} \left| \sum_{n=N+1}^{N+K} a_n \right| < \varepsilon$$

By taking ε smaller, we force N to get larger, so in particular

$$\boxed{\lim_{N \to \infty} \sum_{n=N}^{N} |a_n| = \lim_{N \to \infty} |a_N| = 0}$$

(2) If $\sum_{n=1}^{\infty} a_n$ converges to s, then $s_n \to s$, and so the sequence is bounded, i.e. all s_N satisfy

$$\boxed{\left|\sum_{n=1}^{N} a_n\right| = |s_N| \leq M} \quad \text{for all } N!$$

for some $M > 0$.

(3) Suppose $\sum_{n=1}^{\infty} a_n = s_1$, $\sum_{n=1}^{\infty} b_n = s_2$ are two convergent series. Then so are their sums/differences, products/quotients, etc., & in particular

by limit laws for sequences (Part I)

exercise: prove this

$$\boxed{\begin{aligned} &\sum_{n=1}^{\infty} ca_n = cs_1 = c\cdot\sum_{n=1}^{\infty} a_n \\ &\sum_{n=1}^{\infty} (a_n \pm b_n) = s_1 \pm s_2 = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n \end{aligned}}$$

"summable sequences"

That is,

$$\boxed{(a_n)_{n\in\mathbb{N}},\ (b_n)_{n\in\mathbb{N}} \in \ell \implies (ca_n)_{n\in\mathbb{N}},\ (a_n \pm b_n)_{n\in\mathbb{N}} \in \ell}$$

Making ℓ a vector space!

Recall, $\ell = \{(a_n)_{n\in\mathbb{N}} \mid \sum_{n=1}^{\infty} a_n = s \text{ exists}\} = \{(a_n)_{n\in\mathbb{N}} \mid (s_n)_{n\in\mathbb{N}} \in c\}$

(4) Suppose we don't know if $\sum_{n=1}^{\infty} a_n$ converges,

("monotone series theorem")

but we happen to know the sequence of partial sums is bounded,

$$|s_N| = \left|\sum_{n=1}^{N} a_n\right| \leq M \quad \text{for all } N$$

and either increasing or decreasing. Then the monotone sequence theorem applies to give convergence.

example: $e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$

The partial sums $s_N = \sum_{n=0}^{N} \frac{1}{n!}$ are all bounded by 3,

$$|s_N| = s_N = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{N!}$$

$$\leq 1 + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^N}\right)$$

$$= 1 + \frac{1 - \frac{1}{2^{N+1}}}{1 - \frac{1}{2}}$$

$$\leq 1 + \frac{1}{1 - \frac{1}{2}}$$

$$= 1 + 2 = 3 \quad \text{for all } N$$

because, e.g.

$\frac{1}{3!} = \frac{1}{3} \cdot \frac{1}{2}$

$\leq \frac{1}{2} \cdot \frac{1}{2}$ (bec. $3 \geq 2$)

$\frac{1}{4!} = \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{2}$

$\leq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$

and clearly $s_{N+1} = s_N + \frac{1}{(N+1)!} \geq s_N$, so the seq. s_N is increasing. Therefore, s_N converges, say to e,

$$e = \lim_{N\to\infty} s_N = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{1}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!}$$

QED

(1) Divergence Test: This is the first test you should run, before trying anything else.

$$\boxed{\text{If } \lim_{n\to\infty} a_n \text{ DNE or } \neq 0, \ \sum_{n=1}^{\infty} a_n \text{ is divergent.}}$$

Pf.: By basic property (1), if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$, via an application of the squeeze thm. on $-|a_n| \leq a_n \leq |a_n|$.

(2) Comparison Test (Direct, not Limit)

Two series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$, with

$$\boxed{0 \leq a_n \leq b_n}$$

satisfy:

$$\boxed{\begin{array}{l} (1)\ \sum_{n=1}^{\infty} b_n \text{ convergent} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ convergent} \\ (2)\ \sum_{n=1}^{\infty} a_n \text{ divergent} \Rightarrow \sum_{n=1}^{\infty} b_n \text{ divergent} \end{array}}$$

((4) on p. 5.)

pf: By ~~[illegible]~~ Monotone Series thm.: (1) $0 \le a_n \le b_n$ and $\sum_n b_n$ converges $\Longrightarrow$

$$0 \le s_N \stackrel{\text{def}}{=} \sum_{n=1}^{N} a_n \le t_N \stackrel{\text{def}}{=} \sum_{n=1}^{N} b_n \text{ and } t_N \to \underset{\text{convergent}}{\sum_n b_n}$$

$$\Longrightarrow \quad 0 \le s_N \le \underbrace{\sum_n b_n}_{\text{a pos. \#}}$$

So s_N is bounded,

and clearly increasing

bec. $a_n \ge 0$ for all n

$\Longrightarrow$ $s_N \to s$ (converges)

i.e. $\sum_{n=1}^{\infty} a_n$ converges.

(2) $0 \le a_n \le b_n$ & $\sum_n a_n$ diverges (to ∞) $\Longrightarrow$ $s_N = \sum_{n=1}^{N} a_n$ diverges to ∞ $\Longrightarrow$ $t_N \stackrel{\text{def}}{=} \sum_{n=1}^{N} b_n \ge s_N$ must also diverge to ∞.

QED

(3) <u>Absolute Convergence Implies Convergence</u>:

<u>Def.</u> A series $\sum_{n=1}^{\infty} a_n$ is said to <u>converge absolutely</u> if its associated series $\sum_{n=1}^{\infty} |a_n|$ ("absolute" fr. abs. val.) converges.

$$\boxed{\text{Thm. If } \sum_{n=1}^{\infty} a_n \text{ converges absolutely, then it converges.}}$$

<u>pf:</u> If $\sum_{n=1}^{\infty} |a_n|$ converges, then it's limit, call it T, is a pos. #, so

$$-|a_n| \le a_n \le |a_n|$$

by Comparison Test

$$\Longrightarrow \underbrace{-\sum_{n=1}^{\infty} |a_n|}_{= \sum_{n=1}^{\infty} -|a_n|} \le \sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty} |a_n|$$

by rule (3), p. 4

$$\Longrightarrow -T \le \sum_{n=1}^{\infty} a_n \le T$$

$$\Longrightarrow |S_N| = \left|\sum_{n=1}^{N} a_n\right| \le \left|\sum_{n=1}^{\infty} a_n\right| \le T$$ for all N

so the sequence of partial sums is bounded. Unfortunately we have no idea whether $\sum_{n=1}^{N} a_n = s_N$ is increasing or decreasing, so we cannot possibly use the monotone series theorem. We must fall back on the basics: the error-definition of convergence:

$\sum_{n=1}^{\infty} a_n = s$ converges $\iff$ for any error $\varepsilon > 0$ whatsoever, we can guarantee $\boxed{|s_n - s| < \varepsilon}$ so long as $n \geq N$ (for some N depending on ε)

A basic fact about the real numbers is that (called the Cauchy criterion)

$$\boxed{a_n \longrightarrow a \iff |a_n - a_m| \longrightarrow 0 \quad \text{as } n, m \to \infty}$$

This is precisely what happens here with s_n:

Suppose $n < m$, say $m = n+k$ (otherwise reverse their roles — the case $n=m$ is boring): then

$$
\begin{aligned}
|s_m - s_n| &= \left| \sum_{l=1}^{m} a_l - \sum_{l=1}^{n} a_l \right| \\
&= \left| \sum_{l=1}^{n+k} a_l - \sum_{l=1}^{n} a_l \right| \\
&= \left| \cancel{\sum_{l=1}^{n} a_l} + \sum_{l=n+1}^{n+k} a_l - \cancel{\sum_{l=1}^{n} a_l} \right| \\
&= \left| \sum_{l=n+1}^{n+k} a_l \right| \\
&\le \sum_{l=n+1}^{n+k} |a_l| \longrightarrow 0
\end{aligned}
$$

(because this equals $\big| |s_m| - |s_n| \big| \longrightarrow 0$ since $\sum_{n=1}^{\infty} |a_n|$ converges)

$|s_m| \overset{def}{=} \sum_{k=1}^{m} |a_k|$

Therefore $s_n \longrightarrow s$ for some $s = \sum_{n=1}^{\infty} a_n$. QED

(4) <u>Limit Comparison Test</u>: The comparison test is easy & intuitive, and allows us to demonstrate, for example, that

$\sum_{n=2}^{\infty} \frac{1}{n-1}$ diverges (bec. $\sum_{n=1}^{\infty} \frac{1}{n}$ does, & $\frac{1}{n-1} \geq \frac{1}{n}$ for $n \geq 2$)

&

$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges (bec. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ does & $\frac{1}{n^2+1} \leq \frac{1}{n^2}$)

But we run into difficulties by simple modifications:

$\sum_{n=1}^{\infty} \frac{1}{n+1}$ also diverges, but not by comparison with $\sum_{n=1}^{n} \frac{1}{n}$ since $\frac{1}{n+1} \leq \frac{1}{n}$ (not $\frac{1}{n+1} \geq \frac{1}{n}$, which is what we'd need)

&

$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ ~~al~~also converges, but not by comparison w/ $\sum_{n=1}^{\infty} \frac{1}{n^2}$, since $\frac{1}{n^2-1} \geq \frac{1}{n^2}$ (not $\frac{1}{n^2-1} \leq \frac{1}{n^2}$ which we'd need)

Discussion

To avoid problems relating to the inequalities

$$0 \le a_n \le b_n \quad (\text{\& } -b_n \le -a_n \le 0)$$

which may not hold, we use instead the limit comparison test:

The test:

Limit Comparison Test: Two series with positive terms

$$\boxed{\sum_{n=1}^{\infty} a_n, \quad \sum_{n=1}^{\infty} b_n, \quad a_n, b_n \ge 0}$$

satisfying

$$\boxed{\underbrace{\lim_{n\to\infty} \frac{a_n}{b_n}}_{\text{limit of quotient exists}} = \underbrace{c > 0}_{\text{and is positive}}}$$

both converge or both diverge.

Proof of the limit comp. test

pf: Suppose $\sum_{n=1}^{\infty} b_n = B$ is convergent &

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0 \qquad (*)$$

We want to show that $\sum_{n=1}^{\infty} a_n = A$ converges too.

By the error-formulation of limit (see part I), the limit $(*)$ means: for any error size $\varepsilon > 0$ there is an N after which, i.e. for $n \geq N$, we have

$$\left|\frac{a_n}{b_n} - c\right| < \varepsilon$$

$$\iff \quad -\varepsilon < \frac{a_n}{b_n} - c < \varepsilon$$

$$\iff \quad c - \varepsilon < \underbrace{\frac{a_n}{b_n} < c + \varepsilon}_{\text{use this one here}} \qquad (\text{for } n \geq N)$$

Thus, after passing N, we have

$$0 \leq a_n < (c+\varepsilon) b_n \qquad (n \geq N)$$

$\Longrightarrow$ we can apply the Comparison Test on the tails: $\sum_{n=N+1}^{\infty} a_n = A'$ converges bec. $\underbrace{\sum_{n=N+1}^{\infty} b_n}$ does − N terms $b_1, \ldots$

a finite sum, so a #.

$\Longrightarrow \sum^{\infty} a_n = \underbrace{\sum^{N} a_n} + \sum^{\infty} a_n = A$ converges bec. this $= B$

the other half: Suppose $\sum_{n=1}^{\infty} b_n$ diverges &

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$$

Then, for any error size $\varepsilon > 0$ there is an N s.t. for $n \geq N$ we have

$$\left| \frac{a_n}{b_n} - c \right| < \varepsilon$$

$$\iff \underbrace{c - \varepsilon < \frac{a_n}{b_n}}_{\text{use this one}} < c + \varepsilon$$

Since $c > 0$, we may choose $\varepsilon < \frac{c}{2}$ so that $c - \varepsilon > c - \frac{c}{2} = \frac{c}{2} > 0$, & then use the 1st inequality,

$$\implies b_n (c - \varepsilon) < a_n \quad (n \geq N)$$

so by the comparison test, since $\sum_{n=1}^{\infty} b_n$ diverges (to ∞) (which means $\sum_{n=N+1}^{\infty} b_n$ diverges (to ∞)), so, too, does $\sum_{n=N+1}^{\infty} a_n$, & therefore also $\sum_{n=1}^{\infty} a_n$. <u>QED</u>

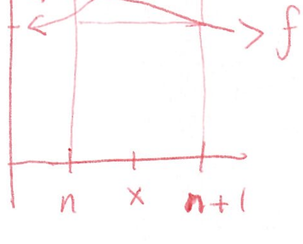
(5) <u>Integral Test</u> Suppose $\sum_{n=1}^{\infty} a_n$ satisfies

$$\boxed{\begin{array}{l} \text{(i) } a_n \geq 0 \\ \text{(ii) } a_n = f(n) \text{ for a } \underline{\text{continuous}}, \underline{\text{decreasing}} \\ \qquad \text{function } f : [1,\infty) \to (0,\infty) \end{array}}$$

Then <u>both $\sum_{n=1}^{\infty} a_n$ & $\int_1^{\infty} f(x)\,dx$ converge or</u> <u>both diverge</u>. <u>In the convergent</u> case, moreover, we have

$$\boxed{\sum_{n=2}^{\infty} a_n < \int_1^{\infty} f(x)\,dx < \sum_{n=1}^{\infty} a_n}$$

<u>pf:</u> For all $n \leq x \leq n-1$, the fact that f is decreasing implies



$$a_n = f(n) \geq f(x) \geq f(n+1) = a_{n+1}$$

$$\Longrightarrow \quad a_n \geq \int_n^{n+1} f(x) \geq a_{n+1}$$

$$\Rightarrow \quad a_2 \le \int_1^2 f(x)\,dx \le a_1$$

$$a_3 \le \int_2^3 f(x)\,dx \le a_2$$

$$\vdots$$

$$\Rightarrow \quad \sum_{k=2}^{n+1} a_k \le \sum_{k=2}^{n+1} \int_{k-1}^{k} f(x)\,dx \le \sum_{k=1}^{n} a_k$$

$$\left(= \int_1^{n+1} f(x) \right)$$

$$\Rightarrow \quad \boxed{s_{n+1} - s_1 \le \int_1^{n+1} f(x)\,dx \le s_n}$$

Thus, $\sum_{n=1}^{\infty} a_n$ converges $\Longleftrightarrow$ s_n converges

$\Longleftrightarrow$ $\int_1^{n+1} f(x)\,dx \longrightarrow \int_1^{\infty} f(x)\,dx$

converges (by Sq. Thm.)

& by the same token (by comparison with the series $\sum_{n=1}^{\infty} b_n$, $b_n = \int_n^{n+1} f(x)\,dx$) diverges $\Longleftrightarrow$ $\int_1^{\infty} f(x)\,dx$ diverges.

QED

(6) <u>p-Series Test</u>

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \begin{array}{l} \text{converges if } p>1 \\ \text{diverges if } p \le 1 \end{array}}$$

pf: By the Integral Test: $f(x) = \frac{1}{x^p}$ is continuous & decreasing on $[1, \infty)$, bec. $f'(x) = -p\frac{1}{x^{p+1}} < 0$, and

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b x^{-p}\,dx$$

$$= \lim_{b\to\infty} \left[\frac{1}{-p+1} x^{-p+1}\right]_1^b \quad \text{if } p \neq 1$$

or

$$\lim_{b\to\infty} \left[\ln x\right]_1^b \quad \text{if } p = 1$$

$$= \lim_{b\to\infty} \left[\frac{1}{-p+1} - \frac{1}{-p+1} b^{-p+1}\right], \quad p \neq 1$$

or

$$\lim_{b\to\infty} \ln b \quad \text{if } p = 1$$

which exists only when $p > 1$.

QED

(7) Geometric Series

$$\boxed{\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \cdots = \frac{1}{1-r}}$$

$\boxed{\text{converges} \iff |r| < 1.}$

Pf:
$$s_N = \sum_{n=0}^{N} r^n = 1 + r + r^2 + \cdots + r^N$$
$$= \frac{1 - r^{N+1}}{1-r},$$

(bec. $(1-r)(1+r+r^2+\cdots+r^N) = 1 + \cancel{r} + \cancel{r^2} + \cdots + \cancel{r^N} - \cancel{r} - \cancel{r^2} - \cdots - \cancel{r^N} - r^{N+1}$
$= 1 - r^{N+1}$)

and $\lim_{N\to\infty} s_N = \frac{1}{1-r}$ exists if & only if
$\lim_{N\to\infty} r^{N+1}$ exists, & this happens $\iff |r| < 1.$

QED