

Ex. 2.7.2

$$(a) \quad 0 \leq \frac{1}{2^{n+1}} \leq \frac{1}{2^n} \quad \& \quad \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = 2$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^n} = 2 - 1 = 1$$

is convergent

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \text{ is convergent}$$

by the Comparison Test.

Ex. 2.7.2

$$(d) \sum_{k=0}^{\infty} \left( \frac{1}{3k+1} + \frac{1}{3k+2} - \frac{1}{3k+3} \right)$$

$$= \sum_{k=0}^{\infty} \left( \frac{\cancel{3k+1} + (3k+2)}{(3k+1)(3k+2)} - \frac{1}{3k+3} \right)$$

$$= \sum_{k=0}^{\infty} \frac{((3k+1) + (3k+2))(3k+3) - (3k+1)(3k+2)}{(3k+1)(3k+2)(3k+3)}$$

$$= \sum_{k=0}^{\infty} \frac{(3k+1)(3k+3) + (3k+2)(3k+3) - (3k+1)(3k+2)}{(3k+1)(3k+2)(3k+3)}$$

*factor out*

$$= \sum_{k=0}^{\infty} \frac{(3k+1)(3k+3) + (3k+2)(\overbrace{(3k+3) - (3k+1)}^{=2})}{(3k+1)(3k+2)(3k+3)}$$

$$= \sum_{k=0}^{\infty} \frac{\cancel{(3k+1)}\cancel{(3k+3)}}{\cancel{(3k+1)}(3k+2)\cancel{(3k+3)}} + \frac{2\cancel{(3k+2)}}{(3k+1)\cancel{(3k+2)}(3k+3)}$$

$$= \sum_{k=0}^{\infty} \frac{1}{3k+2} + \frac{2}{(3k+1)(3k+3)}$$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{1}{3k+2}$$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{1}{3k+3} = \frac{1}{3} \sum_{k=1}^{\infty} \frac{1}{k} = \infty$$

Ex. 2.7.2

$$(e) \quad 1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \dots$$

$$\underbrace{\quad} = \frac{2^2-1}{2^2 \cdot 2} \quad \underbrace{\quad} = \frac{4^2-3}{4^2 \cdot 3} \quad \underbrace{\quad} = \frac{6^2-5}{6^2 \cdot 5}$$

$$= \frac{3}{4}$$

$$= \frac{13}{16 \cdot 3}$$

$$= \frac{31}{36 \cdot 5}$$

$$= \frac{13}{48}$$

$$= \frac{31}{180}$$

and in general

$$\frac{1}{2k+1} - \frac{1}{(2(k+1))^2} = \frac{4(k+1)^2 - (2k+1)}{4(k+1)^2(2k+1)}$$

$$= \frac{4k^2 + 8k + 4 - 2k - 1}{8k^3 + 17k^2 + 16k + 4}$$

$\hookrightarrow -2k - 1 \geq 2$

~~roots of  $4x^2 - 2x - 1$~~

~~are  $x = \frac{2 \pm \sqrt{4+16}}{8}$~~

~~$= \frac{2 \pm 2\sqrt{5}}{8}$~~

~~$\& \frac{2+2\sqrt{5}}{8} \approx 0.81$~~

$$\geq \frac{4k^2 - 2k - 1}{(8+17+16+4)k^3}$$

$$= \frac{4k^2 - 2k - 1}{45k^3}$$

$$\geq \frac{4k^2}{45k^3}$$

$$= \frac{4}{45} \cdot \frac{1}{k}$$

so diverges by Comparison Test.

Ex. 2.7.9 (Ratio Test)

Given  $\sum_{n=1}^{\infty} a_n$  with  $a_n \neq 0$  and with

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = r$$

If  $r < 1$ , the series converges absolutely, while if  $r > 1$ , the series diverges. If  $r = 1$ , no information is given.

(a) ~~pf:~~ Since  $\frac{a_{n+1}}{a_n} \rightarrow r$ , ~~if~~ if  $r < 1$ , then ~~if~~

$$\forall s \in (r, 1), \exists N \in \mathbb{N} \text{ st. } n \geq N$$

$$\implies \frac{a_{n+1}}{a_n} < s$$

because if we choose  $\epsilon = \frac{s-r}{2}$ ,  $\exists N \in \mathbb{N}$  st.

$$n \geq N \implies \left| \frac{a_{n+1}}{a_n} - r \right| < \epsilon \implies \frac{|a_{n+1}|}{|a_n|} < \epsilon + r = s - r + r = s$$

by reverse triangle ineq.

Therefore,

(a)  $n \geq N \Rightarrow |a_{n+1}| < s |a_n|$   
 $< s^2 |a_{n-1}|$   
~~...~~  
 $< s^{n-N+1} |a_N|$   
 $= s^n \cdot \frac{|a_N|}{s^{N-1}}$

(b) But  $0 < s < r < 1 \Rightarrow \frac{|a_N|}{s^{N-1}} \sum_{n=0}^{\infty} s^n = \frac{|a_N|}{s^{N-1}} \cdot \frac{s}{1-s}$   
 converges

(c) and

~~...~~  
 $\sum_{n=1}^m |a_n| \leq \sum_{n=1}^{N-1} |a_n| + \sum_{n=N}^{\infty} |a_n|$   
 $\leq \sum_{n=1}^{N-1} |a_n| + \frac{|a_N|}{s^{N-1}} \cdot \frac{s}{1-s}$

So the partial sums  $S_m = \sum_{n=1}^m |a_n|$  are bounded

and increasing.  
converges, i.e.

The RCT says  $s_m \rightarrow s$   
 $\sum_{n=1}^{\infty} a_n$  is absolutely convergent.