Ex. 2.4.4 (a) M.C.T. $\Rightarrow$ A.P. (Arch. Prop.)

pf: Suppose M.C.T. holds (every bdd. monotone seq. converges), and let

$$x \in \mathbb{R}$$

If $x \le 0$, any $n \ge 1 > 0$ will do, since

$$x \le 0 < 1 \le n$$

$$\Rightarrow x < n$$

So suppose $x > 0$. If

$$n \le x$$

for all $n \in \mathbb{N}$, the seq.

$$a_n = n$$

is bounded (by x), & increasing, so

MCT $\Rightarrow \lim_{n \to \infty} n = L$ exists in $\mathbb{R}$.

Now, choosing $\varepsilon = 1$, $\exists N \in \mathbb{N}$ s.t.

$$n \geq N \Rightarrow |n - L| < 1$$

$$\Rightarrow -1 < n - L < 1$$

$$\Rightarrow L - 1 < n < L + 1$$

Then

$$L - 1 < \underbrace{n + 1 < L + 1}$$ too

$$\Rightarrow n < L$$

&

$$L - 1 < \underbrace{n + 2 < L + 1}$$

$$\Rightarrow n < L - 1$$

contradiction.

Thus, $\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow n > x$.

(b) M.C.T. $\Rightarrow$ N.I.P.

pf: Suppose MCT, that every bdd. monotone seq. converges, & let us show that every nested seq. of closed intervals

$$I_1 \supseteq I_2 \supseteq \cdots \qquad (1)$$

has nonempty intersection,

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset \qquad (2)$$

Let

$$I_n = [a_n, b_n]$$

and rephrase (1) as

$$a_1 \leq a_2 \leq \cdots \quad \leq \cdots b_2 \leq b_1 \qquad (3)$$

The a_n sequence is increasing and bounded above (by any b_m), so by MCT

$$\lim_{n\to\infty} a_n = a$$

exists. Similarly, the b_n sequence is decreasing and bdd. below (by any a_m), so by MCT

$$\lim_{n\to\infty} b_n = b$$

exists. Moreover,

$$a_1 \le \cdots \le a_n \le \cdots \le a \le \cdots \le b_n \le \cdots \le b_1,$$

so

$$a, b \in [a_n, b_n], \quad \forall n \in \mathbb{N}$$

$$\Longrightarrow a, b \in \bigcap_{n=1}^{\infty} I_n,$$ which is therefore nonempty. ✓

NIP ⇒ AoC

Ex 2.5.4 Supp. NIP, that if I_n is a nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq \cdots$$

then

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

and suppose also

$$\frac{1}{2^n} \to 0$$

} this means suppose the Archimedean property

Now let $A \subseteq \mathbb{R}$ be nonempty & bounded above, so

set of upper bds of A $= A^u \neq \emptyset$

Since $A \neq \emptyset$, $\exists a \in A$, & since $A^u \neq \emptyset$, $\exists b \in A^u$, &

$$a \leq b$$

by def. of A^u. Let $a_1 = a$, $b_1 = b$

$$I_1 = [a_1, b_1]$$

Let $c_1 = \frac{a_1+b_1}{2}$ (midpt of I_1) & check whether

$$c_1 \in A^u \text{ or } c_1 \notin A^u$$

If $c_1 \in A^u$, let

$$a_2 = a_1$$
$$b_2 = c_1$$

else let

$$a_2 = c_1$$
$$b_2 = b_1$$

& call

$$I_2 = [a_2, b_2] \subseteq [a_1, b_1] = I_1$$

Repeat: then

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

and

our other assump.

$$\lim_{n\to\infty} (b_n - a_n) = \lim_{n\to\infty} \frac{1}{2^n}(b_1 - a_1) = 0$$

$$\Rightarrow \bigcap_{n=1}^{\infty} I_n = \{a\} \text{ (a single pt.)}$$

Now, $a \in I_n$, $\forall n \in \mathbb{N}$, so

$$a_n \leq a \leq b_n \, , \quad \forall n \in \mathbb{N}$$

$\Rightarrow$ a is an upper bound for all the $a_n \in A$, but $a \leq b_n$, so it is smaller than all $b_n \in A^u$, i.e.

$$a = \min A^u =: \sup A$$

NIP $\Longrightarrow$ B.W.

Ex. 2.5.9 Let $S = \{x \in \mathbb{R} \mid x < a_n$ for infinitely many $a_n\}$

for a bounded seq. $(a_n)_{n\in\mathbb{N}}$. Since $(a_n)_{n\in\mathbb{N}}$ is bdd, $\exists M > 0$ s.t.

$$|a_n| \leq M, \quad \forall n \in \mathbb{N}$$

$$\Longleftrightarrow \quad -M < a_n < M, \quad \forall n \in \mathbb{N}$$

so $-M \in S$ & $S \neq \emptyset$. Let

$$c = \frac{-M + a_1}{2}$$

If $c_1 \in S$, let

$$a_{n_1} = a_1$$

$$a_{n_2} \in \underbrace{[c_1, a_1]}_{= I_1} \text{ in the seq.}$$

else let

$$a_{n_1} = a_1 \qquad = I_1$$

$$a_{n_2} \in [-M, c_1] \text{ in the seq.}$$

Repeat. Then,

$$a_{n_{k+1}} \le a_{n_k}$$

and since $a_{n_k} \in I_k \overset{= [\alpha_k, \beta_k]}{}$ &

$$\lim_{k\to\infty} (\beta_k - \alpha_k) = \lim_{k\to\infty} \frac{1}{2^k}\cdot 2M = 0$$

& $\displaystyle\bigcap_{k=1}^{\infty} I_k \ne \emptyset$

we have

$$\bigcap_{k=1}^{\infty} I_k = \{a\}$$

and

$$\lim_{k\to\infty} a_{n_k} = a$$

(Squeeze thm. on

$\alpha_k \le a_{n_k} \le \beta_k$

since $\alpha_k \to a$

$\beta_k \to a$)