Ex. 2.6.7 (a) – (b)

Thm. The following are equivalent:

(1) A.o.C. (Axiom of Completeness) (every nonempty $S \subseteq \mathbb{R}$ bdd. above has a least upper bound)

$\sup S := \min S^u$

(2) M.C.T. (Monotone Convergence Thm.)

(every bdd. monotone sequence of real numbers converges)

(3) N.I.P. (Nested Interval Property)

(every nested sequence of closed intervals $I_1 \supseteq I_2 \supseteq \cdots$ has nonempty intersection, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$)

(4) <u>B.W.</u> (<u>Bolzano – Weierstrass Thm.</u>)

(every bdd. sequence has a convergent subsequence)

(5) <u>C.C.</u> (<u>Cauchy Criterion</u>)

(every real sequence converges $\iff$ it is Cauchy)

<u>pf:</u> The implications

AoC $\Rightarrow$ NIP (Thm. 1.4.1) (i)

NIP $\Rightarrow$ MCT (Thm. 2.4.2) (ii)

have already been dealt with, as have

MCT $\Rightarrow$ NIP (Ex. 2.4.4 (b)) (iii)

NIP $\Rightarrow$ AoC (Ex. 2.5.4) (iv)

NIP $\iff$ MCT i.e. (2) $\iff$ (3)

AoC $\iff$ NIP, i.e. (1) $\iff$ (3)

Thus, we have the equivalence of

$$AoC \Longleftrightarrow NIP \Longleftrightarrow MCT$$

(i), (iv) (ii), (iii)

or

$$(1) \Longleftrightarrow (2) \Longleftrightarrow (3)$$

Also, we saw

$$NIP \Longrightarrow B.W$$ (Thm. 2.5.5) (v.)

and

$$BW \Longrightarrow CC$$ (Thm. 2.6.4) (vi.)

or

$$(3) \Longrightarrow (4) \Longrightarrow (5)$$

Thus, the exercise will be complete if we prove $(5) \Longrightarrow (3)$, i.e. $CC \Longrightarrow NIP$:

Suppose, therefore, that CC holds (that a sequence converges $\iff$ it is Cauchy), & consider a nested chain

$$I_1 \supseteq I_2 \supseteq \cdots$$

of closed intervals $I_n = [a_n, b_n]$. We want to show $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. There are 2 cases:

(1) $\lim_{n\to\infty} (b_n - a_n) = 0$

(2) $\lim_{n\to\infty} (b_n - a_n) > 0$

If case (1) holds, the sequence $d_n := b_n - a_n$ is convergent, hence the sequence

$$c_n := \frac{a_n + b_n}{2}$$

satisfying

$$a_n \le c_n \le b_n$$

must also converge, because we can show it is Cauchy: since $d_n = b_n - a_n \to 0$, it is Cauchy: $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t.

$$n, m \ge N \implies |d_n - d_m| < \varepsilon$$

WOLOG suppose $n \ge m \ge N$

$$\implies |(b_n - a_n) - (b_m - a_m)| < \varepsilon$$

$$\implies |c_n - c_m| = \left| \frac{b_n + a_n}{2} - \frac{b_m + a_m}{2} \right|$$

$$= \frac{1}{2} |(b_n - b_m) + (a_n - a_m)|$$

By the reverse triangle inequality we have

$$||b_n - a_n| - |b_m - a_m|| \le |(b_n - a_n) - (b_m - a_n)|$$

satisfying

$$a_n \le c_n \le b_n$$

must also converge, because it is Cauchy, as we now show: since

$$d_n := b_n - a_n \longrightarrow 0$$

we have, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$$n \ge N \implies |b_n - a_n| = |d_n - 0| < \varepsilon$$

so

$$n, m \ge N \implies |c_n - c_m| = \left|\frac{(a_n + b_n)}{2} - \frac{a_m + b_m}{2}\right|$$

$$= \frac{1}{2}\left|(a_n - a_m) + (b_n - b_m)\right|$$

$$\le \frac{1}{2}|a_n - a_m| + \frac{1}{2}|b_n - b_m|$$

and since, WOLOG, we may suppose $n \ge m \ge N$,

$$a_n \le a_m \le b_n \implies |a_n - a_m| \le |a_n - b_n| < \varepsilon$$

& since w/ $|b_n - b_m| \le |b_n - a_n| < \varepsilon$

we have

$$|c_n - c_m| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

Thus, c_n is Cauchy, & CC implies

$$a_n \leq c = \lim_{n \to \infty} c_n \leq b_n$$

exists in $\mathbb{R}$, & moreover

$$c \in \bigcap_{n=1}^{\infty} I_n$$

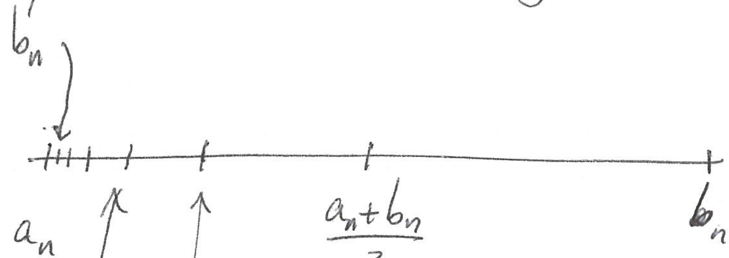
by the line above. This completes case (1), when $\lim_{n \to \infty} (b_n - a_n) = 0$.

Case (2), $\lim_{n \to \infty} b_n - a_n > 0$ can be reduced to case (1) after we come to understand why Case (1) must even possess the limit $\lim_{n \to \infty} b_n - a_n$.

Consider the sequence

$$b_n' := \frac{a_n + b_n}{2^n} + \frac{a_n}{2^{n-1}} + \frac{a_n}{2^{n-2}} + \cdots + \frac{a_n}{2}$$

Gotten by repeatedly halving n times:



$$\frac{a_n + \left(\frac{a_n+b_n}{2}\right)}{2} = \frac{a_n}{2} + \frac{a_n+b_n}{2^2}$$

$$\frac{a_n + \left(\frac{a_n}{2} + \frac{a_n+b_n}{2^2}\right)}{2} = \frac{a_n}{2} + \frac{a_n}{2^2} + \frac{a_n+b_n}{2^3}$$

Then, with $a_n' = a_n$, $b_n' = \sum_{k=1}^{n-1} \frac{a_n}{2^k} + \frac{a_n+b_n}{2^n}$

we have

$$|b_n' - a_n'| = \left| \sum_{k=1}^{n-1} \frac{a_n}{2^k} + \frac{a_n+b_n}{2^n} - a_n \right|$$

$$= \frac{1}{2^n}(b_n - a_n) \leq \frac{1}{2^n}(b_1 - a_1)$$

Since $\overbrace{\frac{1}{2^n} = \left(\frac{1}{2}\right)^n \longrightarrow 0 \text{ bc. } |1/2| < 1}^{\text{Requires Archimedean Property}}$ (Ex. 2.5.7), the algebraic limit laws show that $\frac{1}{2^n}(b_1 - a_1) \longrightarrow 0$, too, & hence, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$$n \geq N \implies |b_n' - a_n'| \leq \frac{1}{2^n}(b_1 - a_1) < \varepsilon$$

Thus,

$$\lim_{n\to\infty} (b_n' - a_n') = 0 \qquad (*)$$

and by Case (1)

$$\boxed{\begin{aligned} \emptyset \neq \bigcap_{n=1}^{\infty} [a_n', b_n'] \\ \subseteq \bigcap_{n=1}^{\infty} [a_n, b_n] \end{aligned}}$$

bc $[a_n', b_n'] \subseteq [a_n, b_n]$ $\forall n \in \mathbb{N}$.

Moreover, by $(*)$, $\bigcap_{n=1}^{\infty} [a_n', b_n'] = \{a\}$

and we conclude that

$$\boxed{\lim_{n\to\infty} a_n = a} \in \{a\} = \bigcap_{n=1}^{\infty} [a_n', b_n']$$

bec.

$$|a_n - a| \le |a_n - b_n'| = |a_n' - b_n'|$$

can be made $< \varepsilon$ for large enough n (by $(b_n' - a_n') \to 0$).

A similar argument shows that

$$\lim_{n\to\infty} b_n = b \quad \text{exists}$$

and by the order limit laws we know

$$a_n \le a \le b \le b_n, \quad \forall n \in \mathbb{N}$$

so

$$\emptyset \neq [a,b] \subseteq \bigcap_{n=1}^{\infty} I_n$$

$=$ in fact

QED