Sequences of Real Numbers, Part III

Infinite Series – Alternating Series and Ratio Test

Def. 1 An alternating series has terms whose sign switches every n,

$$\boxed{\sum_{n=1}^{\infty} (-1)^{n-1} b_n \quad \text{where all } b_n > 0}$$

i.e.

$$\sum_{n=1}^{\infty} a_n \quad \text{where } a_n = (-1)^{n-1} b_n,\ b_n > 0$$

($(-1)^{n-1}$ isn't strictly necessary, we could have $(-1)^n$ too)

Ex. 1 $$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$ (alternating harmonic series)

Ex. 2 $$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1} = -\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \cdots$$

Ex. 3 $$\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1} = -3 + \frac{6}{7} - \frac{9}{11} + \cdots$$

Def. 2 We recall that a series $\sum_{n=1}^{\infty} a_n$ is called <u>absolutely convergent</u> if

$$\sum_{n=1}^{\infty} |a_n|$$

is convergent, and that by a theorem in Part II (p. 9) in this case $\sum_{n=1}^{\infty} a_n$ converges as well. But <u>alternating series provide examples of series which are convergent, but not absolutely convergent</u>! This is why we must introduce a <u>middle term</u>:

$$\sum_{n=1}^{\infty} a_n$$

is called <u>conditionally convergent</u> if <u>it is convergent, but not absolutely convergent</u>.

<u>Thm. 1</u> <u>(Alternating Series Test)</u> If an alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n, \quad (b_n > 0)$$

satisfies

(i.e., <u>eventually</u> <u>$b_n \searrow 0$</u>) $\left\{ \boxed{\begin{array}{l} \text{(i)} \quad b_n \geq b_{n+1} \quad \text{for all } n \\ \text{(ii)} \quad \lim_{n\to\infty} b_n = 0 \end{array}} \right.$ (<u>Remark</u>: all $n \geq N$ for some fixed N)

<u>Then $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges.</u>

<u>pf:</u> Let $\boxed{s_N = \sum_{n=1}^{N} (-1)^{n-1} b_n}$ and suppose for $n \geq N$ we have $b_n \geq b_{n+1}$ ($\&$ $b_n \searrow 0$). Since s_N is a finite number, the essential question is what happens to <u>the tail</u>:

$$\boxed{\sum_{n=N+1}^{\infty} (-1)^{n-1} b_n}$$

Define <u>its</u> kth partial sum, $\boxed{t_k \stackrel{\text{def}}{=} \sum_{n=N+1}^{k} (-1)^{n-1} b_n,}$

and observe that when $k - N$ is even, $k - N = 2l$,

$$
\begin{aligned}
t_k &= \sum_{n=N+1}^{k} (-1)^{n-1} b_n \\
&= (-1)^N b_{N+1} + (-1)^{N+1} b_{N+2} + \cdots + (-1)^k b_k \\
&= (-1)^N \left[b_{N+1} - b_{N+2} + b_{N+3} - \cdots - b_{k-1} + b_k \right] \\
&= (-1)^N \Big[b_{N+1} - \underbrace{(b_{N+2} - b_{N+3})}_{\geq 0} \\
&\qquad - \underbrace{(b_{N+4} - b_{N+5})}_{\geq 0} \\
&\qquad - \cdots - \underbrace{(b_{k-1} - b_k)}_{\geq 0} \Big]
\end{aligned}
$$

(bec. $(-1)^{k-N} b_k = (-1)^{2l} b_k = b_k$)

and therefore

$$
\begin{aligned}
s_k &= s_N + t_k \\
&= s_N + (-1)^N \left[b_{N+1} - (b_{N+2} - b_{N+3}) - \cdots - (b_{k-1} - b_k) \right] \\
&\leq s_N + (-1)^N b_{N+1} \\
&= s_N + b_{N+1}
\end{aligned}
$$

(if N is even, which we now just assume, else replace N with $N+1$)

Thus, the seq. $(s_N)_{N \in \mathbb{N}}$ is bounded. It is also increasing, since $s_{N+3} = s_{N+1} + (b_{N+2} - b_{N+3}) \geq s_{N+1}$.

By the Monotone Seq. thm., s_N must converge,

$$s = \lim_{N \to \infty} s_N = \sum_{n=1}^{\infty} (-1)^{n-1} b_n \qquad \text{QED}$$

Ex. The alternating harmonic series

$$\boxed{\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots}$$

is the archetypal example of a conditionally convergent series, since

(a) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ converges by the alt. series test: $b_n = \frac{1}{n} \searrow 0$ for all $n \geq 1$, and

(b) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ is not absolutely convergent,

bec. $\sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{1}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$

is the regular harmonic series, which diverges.

Thm. 2 (Ratio Test) If the ratios $\frac{a_{n+1}}{a_n}$ of consecutive terms of a series $\sum_{n=1}^{\infty} a_n$ eventually stabilize, i.e.

$$\boxed{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L \quad \text{exists}}$$

then

(1) $L < 1 \implies \sum_{n=1}^{\infty} a_n$ converges absolutely (hence converges)

(2) $L > 1 \implies \sum_{n=1}^{\infty} a_n$ diverges

(3) $L = 1 \implies$ No Info

pf: Too difficult, see any undergrad. analysis book, e.g. Ross, "Elementary Analysis." QED