Ex. 2.3.13

(a) Let $a_{mn} = \frac{m}{m+n}$. Then,

$$\lim_{n\to\infty}\left(\lim_{m\to\infty}\frac{m}{m+n}\right) = \lim_{n\to\infty} 1 = 1$$

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{m}{m+n}\right) = \lim_{m\to\infty} 0 = 0$$

(b) Let $a_{mn} = \frac{1}{m+n}$. Then

$$\lim_{m,n\to\infty} a_{mn} = 0$$

bec. $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t.

$$\frac{1}{N} < 2\varepsilon \quad \text{(by Arch. Prop.)}$$

$$\Rightarrow \forall m,n \geq N,\ \frac{1}{m} \leq \frac{1}{N},\ \frac{1}{n} \leq \frac{1}{N} \Rightarrow$$

$$\frac{1}{m+n} \leq \frac{1}{2N} < \varepsilon$$

$$\Rightarrow |a_{mn} - 0| = \frac{1}{m+n} < \varepsilon$$

Also,

$$\lim_{n\to\infty}\left(\lim_{m\to\infty}\frac{1}{m+n}\right) = 0 = \lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{1}{m+n}\right)$$

ex. where iterated limits exist but not the limit

Now let $a_{mn} = \dfrac{mn}{m^2+n^2}$:

$$\lim_{n\to\infty}\left(\lim_{m\to\infty}\frac{mn}{m^2+n^2}\right) = 0 = \text{vice-versa}$$

and

$$\underline{\lim_{m,n\to\infty}\frac{mn}{m^2+n^2}\ \text{DNE}}$$

bec. for $m=n$,

$$\lim_{m,n\to\infty} a_{mn} = \lim_{n\to\infty}\underline{\frac{n^2}{2n^2} = \frac{1}{2} \neq 0}$$

(c) Let $a_{mn} = (-1)^{n+m}\left(\frac{1}{n}+\frac{1}{m}\right)$

then $\lim\limits_{m\to\infty} a_{mn}$ & $\lim\limits_{n\to\infty} a_{mn}$ DNE

but $\lim\limits_{m,n\to\infty} a_{mn} = 0$ bec $\forall \varepsilon>0, \exists N\in\mathbb{N}$ s.t. $\frac{1}{N} <$

s.t. $m,n \geq N \Rightarrow |a_{mn}-0| = \frac{1}{n}+\frac{1}{m} \leq \frac{2}{N} < \varepsilon.$

(d) Suppose $\lim_{m,n\to\infty} a_{mn} = a$ & $\forall m \in \mathbb{N}$

$\lim_{n\to\infty} a_{mn} = b_m$ exists.

To show $b_m \to a$, let $\varepsilon > 0$ be given. Then, since $a_{mn} \to a$, $\exists N_1 \in \mathbb{N}$ s.t.

$$m, n \geq N_1 \implies |a_{mn} - a| < \varepsilon/2$$

and fixing m here, $a_{mn} \to b_m$ means $\exists N_2 \in \mathbb{N}$ s.t. $n \geq N_2 \implies$

$$|a_{mn} - b_m| < \varepsilon/2$$

Let $N = \max\{N_1, N_2\}$, then $m, n \geq N \implies$

$$\begin{aligned} |b_m - a| &= |b_m - a_{mn} + a_{mn} - a| \\ &\leq |b_m - a_{mn}| + |a_{mn} - a| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

(e) If $\lim_{m,n\to\infty} a_{mn} = a$

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) = b$$

$$\lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right) = c$$

all exist, then by (d) above

$$a=b$$

&

$$a=c$$

bec. $\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) = \lim_{n\to\infty} b_n = b$,

& sim. with the other.

Ex 2.5.4 Supp. NIP, that if I_n is a nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq \cdots$$

then

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

and suppose also

$$\frac{1}{2^n} \to 0$$

Now let $A \subseteq \mathbb{R}$ be nonempty & bounded above, so

$$\text{set of upper bds of } A = A^u \neq \emptyset$$

Since $A \neq \emptyset$, $\exists a \in A$, & since $A^u \neq \emptyset$, $\exists b \in A^u$, &

$$a \leq b$$

by def. of A^u. Let $a_1 = a$, $b_1 = b$

$$I_1 = [a_1, b_1]$$

Let $c_1 = \frac{a_1+b_1}{2}$ (midpt of I_1) & check whether

$$c_1 \in A^u \text{ or } c_1 \notin A^u$$

If $c_1 \in A^u$, let

$$a_2 = a_1$$
$$b_2 = c_1$$

else let

$$a_2 = c_1$$
$$b_2 = b_1$$

& call

$$I_2 = [a_2, b_2] \subseteq [a_1, b_1] = I_1$$

Repeat: then

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

and

our other assump.

$$\lim_{n\to\infty} (b_n - a_n) = \lim_{n\to\infty} \frac{1}{2^n}(b_1 - a_1) = 0$$

$$\Rightarrow \bigcap_{n=1}^{\infty} I_n = \{a\} \text{ (a single pt.)}$$

Now, $a \in I_n$, $\forall n \in \mathbb{N}$, so

$$a_n \leq a \leq b_n \; , \; \forall n \in \mathbb{N}$$

$\implies$ a is an upper bound for all the $a_n \in A$, but $a \leq b_n$, so it is smaller than all $b_n \in A^u$, i.e.

$$a = \min A^u =: \sup A$$

Ex. 2.5.7 $|b|<1 \iff \lim_{n\to\infty} b^n = 0$

pf: (1) $|b|<1 \implies |b|^2 < |b| < 1$

$\implies |b|^{n+1} < |b|^n$

so $|b|^n$ is a monontone decreasing sequence.

Now, $|b|^n > 0$, so by the Monotone Conv. Thm. $\lim_{n\to\infty} |b|^n = a$ exists.

Claim: $\lim_{n\to\infty} |b|^n = 0$: $\forall \varepsilon > 0$,

$\exists N \in \mathbb{N}$ s.t.

$$\frac{1}{N} < \frac{\ln\left(\frac{1}{|b|}\right)}{\ln\left(\frac{1}{\varepsilon}\right)}$$

make sure $\varepsilon < 1$

so $n \geq N \implies \frac{1}{n} \leq \frac{1}{N} < \frac{\ln(1/|b|)}{\ln(1/\varepsilon)}$

$\implies \ln(1/\varepsilon) < n \ln(1/|b|) = \ln\left(\frac{1}{|b|^n}\right)$

$$\Rightarrow \frac{1}{\varepsilon} < \frac{1}{|b|^n}$$

$$\Rightarrow |b|^n < \varepsilon$$

$$\Rightarrow \big||b|^n - 0\big| < \varepsilon$$

Since

$$\lim_{n\to\infty} (-|b|^n) = -\lim_{n\to\infty} |b|^n = 0$$

&

$$-|b|^n \le b^n \le |b|^n$$

the Squeeze thm. gives

$$\lim_{n\to\infty} b^n = 0$$

(2) $\lim_{n\to\infty} b^n = 0 \Rightarrow |b| < 1$

for if $|b| = 1$ then $\lim_{n\to\infty} b^n = 1$ or DNE (if $b = -1$),

while if $|b| > 1$ then $|b|^{n+1} > |b|^n > 1 > 0$

$\Rightarrow \lim_{n\to\infty} b^n = \infty$ or DNE (if $b<0$) $\Rightarrow \lim_{n\to\infty} |b|^n > 0$, contradiction.

Ex. 2.5.9 Let $S = \{x \in \mathbb{R} \mid x < a_n$ for infinitely many $a_n\}$

for a bounded seq. $(a_n)_{n \in \mathbb{N}}$. Since $(a_n)_{n \in \mathbb{N}}$ is bdd, $\exists M > 0$ s.t.

$$|a_n| \leq M, \quad \forall n \in \mathbb{N}$$

$$\iff -M < a_n < M, \quad \forall n \in \mathbb{N}$$

so $-M \in S$ & $S \neq \emptyset$. Let

$$c = \frac{-M + a_1}{2}$$

If $c_1 \in S$, let

$a_{n_1} = a_1$

$a_{n_2} \in \underbrace{[c_1, a_1]}_{= I_1}$ in the seq.

else let

$a_{n_1} = a_1$ $\quad = I_1$

$a_{n_2} \in [-M, c_1]$ in the seq.

Repeat. Then,

$$a_{n_{k+1}} \le a_{n_k}$$

and since $a_{n_k} \in I_k = [\alpha_k, \beta_k]$ &

$$\lim_{k\to\infty} (\beta_k - \alpha_k) = \lim_{k\to\infty} \frac{1}{2^k} \cdot 2M = 0$$

& $$\bigcap_{k=1}^{\infty} I_k \neq \emptyset$$

we have
$$\bigcap_{k=1}^{\infty} I_k = \{a\}$$

and
$$\lim_{k\to\infty} a_{n_k} = a$$

(Squeeze Thm. on

$\alpha_k \le a_{n_k} \le \beta_k$

since $\alpha_k \to a$

$\beta_k \to a$)

Ex. 2.4.4 (a) M.C.T. $\Rightarrow$ A.P.

pf: Suppose M.C.T. holds (every bdd. monotone seq. converges), and let

$x \in \mathbb{R}$

If $x \le 0$, any $n \ge 1 > 0$ will do, since

$$x \le 0 < 1 \le n$$
$$\Rightarrow x < n$$

So suppose $x > 0$. If

$$n \le x$$

for all $n \in \mathbb{N}$, the seq.

$$a_n = n$$

is bounded (by x), & increasing, so

MCT $\Rightarrow \lim_{n \to \infty} n = L$ exists in $\mathbb{R}$.

Now, choosing $\varepsilon = 1$, $\exists N \in \mathbb{N}$ s.t.

$$n \geq N \implies |n - L| < 1$$

$$\implies -1 < n - L < 1$$

$$\implies L - 1 < n < L + 1$$

Then

$$L - 1 < \underbrace{n+1 < L+1} \quad \text{too}$$

$$\implies n < L$$

&

$$L - 1 < \underbrace{n+2 < L+1}$$

$$\implies n < L - 1$$

contradiction.

Thus, $\exists N \in \mathbb{N}$ s.t. $n \geq N \implies n > x$.