Babylonian Method / Heron's Method

Let $\sqrt{a}$ be a root of $p(x) = x^2 - a$, $a > 0$.

Suppose we make an initial estimate / guess,

$$x_0 \approx \sqrt{a} \qquad \text{(initial guess)}$$

and let

$$\varepsilon := \sqrt{a} - x_0 \qquad \text{(error)}$$

$$(\varepsilon \ll x_0)$$

Then,

$$\begin{aligned} a &= (x_0 + \varepsilon)^2 \\ &= x_0^2 + 2x_0\varepsilon + \varepsilon^2 \\ &= x_0^2 + \varepsilon(2x_0 + \varepsilon) \end{aligned}$$

$$\Longrightarrow \quad \varepsilon = \frac{a - x_0^2}{2x_0 + \varepsilon} \approx \frac{a - x_0^2}{2x_0}$$

$$\Longrightarrow \quad x_0 + \varepsilon \approx x_0 + \frac{a - x_0^2}{2x_0 + \cancel{\varepsilon}} = \frac{2x_0^2 + a - x_0^2}{2x_0}$$

$$\frac{2x_0}{2x_0}$$

$$= \frac{1/x_0}{1/x_0} \cdot \frac{x_0^2 + a}{2x_0} = \frac{x_0 + \frac{a}{x_0}}{2}$$

$$= \frac{1}{2}\left(x_0 + \frac{a}{x_0}\right)$$

$$\Longrightarrow \quad x_0 \approx x_0 + \varepsilon \approx \tfrac{1}{2}\left(x_0 + \tfrac{a}{x_0}\right)$$

Now, this points to an idea, namely,

~~perhaps by chance.~~

$$x_0 = \tfrac{1}{2}\left(x_0 + \tfrac{a}{x_0}\right) \iff 2x_0 = x_0 + \frac{a}{x_0}$$

$$\iff x_0 = \frac{a}{x_0}$$

$$\iff x_0^2 = a \quad !$$

i.e. we <u>want</u> $x = \frac{1}{2}(x + \frac{a}{x})$, i.e. we <u>want</u> x to be a fixed-point of

$$g(x) = \tfrac{1}{2}\left(x + \tfrac{a}{x}\right)$$

Observe: $\quad g'(x) = \frac{1}{2} - \frac{a}{2x^2} = \left(1 - \frac{a}{x^2}\right)/2$.

Since $\sqrt{a} \in \mathbb{R} \setminus \mathbb{Q}$, presumably, if $n = \lfloor \sqrt{a} \rfloor$, then

$$n < \sqrt{a} < n+1 \iff n^2 < a < (n+1)^2$$

$$\iff 1 < \frac{a}{n^2} < (n+1)^2/n^2 \;\;\&\;\; \frac{a}{(n+1)^2} < 1$$