7/15/20 (The Euler Number e)

Example: The sequence

$$a_n = \left(1+\frac{1}{n}\right)^n$$

is convergent, with limit

$$\boxed{e \ (\approx 2.71828) = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n}$$

the base of the natural logarithm & exponential ($\ln x$, e^x), and moreover, e also has an infinite series representation,

$$\boxed{e = \sum_{n=0}^{\infty} \frac{1}{n!}}$$

pf: We use the Monotone Convergence Thm., showing that both sequences

$$a_n = \left(1+\frac{1}{n}\right)^n$$

$$s_n = \sum_{k=0}^{n} \frac{1}{k!}$$

are increasing and bounded. We will show

(a) $\quad 0 \le a_n \le s_n < 3$ (boundedness of each sequence)

(b) $\quad a_n \le a_{n+1}$

$\quad s_n \le s_{n+1}$ (monotonicity: each seq. is increasing)

From this we conclude that

$$e \le \sum_{n=0}^{\infty} \frac{1}{n!}$$

so to show equality we show also

(c) $$e \ge \sum_{n=0}^{\infty} \frac{1}{n!}$$

We begin with (a):

When expanding, using Binomial thm.,

$$a_n = \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \qquad (1)$$

$$= \binom{n}{0} + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n^2}$$

$$+ \cdots + \binom{n}{n-1}\frac{1}{n^{n-1}} + \binom{n}{n}\frac{1}{n^n}$$

we observe that

$$\binom{n}{k}\frac{1}{n^k} = \frac{n!}{k!\,(n-k)!} \cdot \frac{1}{n^k} \qquad (2)$$

$$= \frac{\big(n(n-1)\cdots(n-k+1)\big)\cancel{(n-k)!}}{k!\,\cancel{(n-k)!}} \cdot \frac{1}{n^k}$$

$$= \frac{1}{k!} \cdot \frac{n(n-1)\cdots(n-k+1)}{n^k}$$

$$= \frac{1}{k!} \cdot \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-k+1}{n}$$

$$= \frac{1}{k!} \cdot 1 \cdot \underbrace{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)}_{0 < \text{ but } < 1}$$

$$< \frac{1}{k!}$$

$$\frac{n-l}{n} = \frac{n}{n} - \frac{l}{n} = 1 - \frac{l}{n}$$

Using the inequality (2) on (1) gives

$$a_n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k}$$

$$\leq \sum_{k=0}^{n} \frac{1}{k!}$$

$$= s_n$$

Clearly $a_n \geq 0$, so now we prove $s_n < 3$:

$$s_n = \sum_{k=0}^{n} \frac{1}{k!} \leq \left(\sum_{k=1}^{n} \frac{1}{2^{k-1}} \right) + 1$$

$$\leq \left(\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{2^{k-1}} \right) + 1$$

$$= \frac{1}{1 - 1/2} + 1$$

$$= 2 + 1$$

$$= 3$$

because for $k \geq 1$ (k=0 corresponds to the extra 1)

$$\frac{1}{k!} = \frac{1}{k \cdot (k-1) \cdots 3 \cdot 2 \cdot 1}$$

$$= \underbrace{\frac{1}{k} \cdot \frac{1}{k-1} \cdots \frac{1}{3}} \cdot \frac{1}{2} \cdot \frac{1}{1} < \overbrace{\frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2} \cdot \frac{1}{2}}^{= \frac{1}{2^{k-1}}} \cdot 1$$

$\frac{1}{3} < \frac{1}{2}$

$\frac{1}{4} < \frac{1}{2}$

$\cdots \frac{1}{k} < \frac{1}{2}$

This completes (a). Now for (b),

(i) $\quad a_n \le a_{n+1}$

(ii) $\quad s_n \le s_{n+1}$

Well, (ii) is obvious, because

$$
\begin{aligned}
s_n &= \sum_{k=0}^{n} \frac{1}{k!} \\
&\le \sum_{k=0}^{n} \frac{1}{k!} + \underbrace{\frac{1}{(n+1)!}}_{\text{extra term}} \\
&= s_{n+1}
\end{aligned}
$$

For (i), we have to do more work:

<u>Exercise:</u> Let $a_1, \dots, a_n > 0$. Show that if

$$\boxed{\begin{aligned} A_n &:= \frac{a_1 + a_2 + \cdots + a_n}{n} \\ G_n &:= (a_1 \cdot a_2 \cdots a_n)^{1/n} \end{aligned}}$$

are their <u>arithmetic</u> and <u>geometric means</u>, resp., then

$$\boxed{A_n \geq G_n \quad \text{for all } n \in \mathbb{N}}$$

<u>Lemma:</u> $(1+\frac{1}{n})^n < (1+\frac{1}{n+1})^{n+1}$ for all $n \in \mathbb{N}$.

<u>pf:</u> Let $a_1 = 1$

$$a_2 = a_3 = \cdots = a_{n+1} = 1 + \frac{1}{n}$$

& apply $G_{n+1} \leq A_{n+1}$:

$$\left(1+\frac{1}{n}\right)^{n/n+1} = \Big(\underset{a_1}{1}\cdot\overbrace{\underset{a_2}{(1+1/n)}\underset{a_3}{(1+1/n)}\cdots\underset{a_{n+1}}{(1+1/n)}}^{n}\Big)^{1/n+1}$$

$$= G_{n+1}$$

$$\leq A_{n+1}$$

$$= \frac{1+n(1+1/n)}{n+1}$$

$$= \frac{1+\cancel{n}\left(\frac{n+1}{\cancel{n}}\right)}{n+1}$$

$$= \frac{n+2}{n+1}$$

$$= 1+\frac{1}{n+1}$$

$$\Longrightarrow \left(1+\frac{1}{n}\right)^{n} \leq \left(1+\frac{1}{n+1}\right)^{n+1} \quad \checkmark$$

This completes (b) & shows that $a_n \leq a_{n+1}$ &

$s_n \leq s_{n+1}$.

By the Monotone Convergence Thm. both

$e = \lim\limits_{n\to\infty} a_n$ & $\sum\limits_{n=0}^{\infty}\frac{1}{n!}$ exist so

$$0 \le a_n \le s_n \implies$$

$$\boxed{\begin{aligned} e &= \lim_{n\to\infty} a_n \\ &\le \lim_{n\to\infty} s_n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \end{aligned}}$$

The last step, (c), is to show also that

$$e \ge \sum_{n=0}^{\infty} \frac{1}{n!}$$

pf: $m \le n \implies$

$$a_n = \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k}$$

$$\ge \sum_{k=0}^{m} \binom{n}{k} \frac{1}{n^k}$$

$$\overset{(2)\,p.3}{=} \underbrace{\sum_{k=0}^{m} \frac{1}{k!} \underbrace{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)}_{\to 1 \text{ as } n\to\infty}}_{\text{call } b_n}$$

Thus, $m \le n \implies a_n \ge b_n, \ \forall n \in \mathbb{N}$

$$\implies \lim_{n\to\infty} a_n \ge \lim_{n\to\infty} b_n$$

$$\implies e \ge s_m = \sum_{k=0}^{m} \frac{1}{k!}$$

$$\implies e \ge \sum_{n=0}^{\infty} \frac{1}{n!} = \lim_{m\to\infty} s_m$$

QED