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Preliminaries

Prop-1 $\boxed{\text{If } a_n \to a > 0, \text{ then } a_n > 0 \text{ for some } n \in \mathbb{N}}$

Sim., $\boxed{\text{if } a_n \to a < 0, \text{ then } a_n < 0 \text{ for some } n \in \mathbb{N}}$

pf: For $a>0$, let $\varepsilon = a$, & find $N \in \mathbb{N}$ s.t.

$$|a_n - a| < a$$

$$\Rightarrow \underbrace{-a < a_n - a}_{} < a$$

$$\Rightarrow 0 < a_n$$

Sim., if $a<0$, then $|a_n| \to |a| > 0$

$$\Rightarrow |a_n| > 0 \text{ for some } n \in \mathbb{N}$$

$$\Rightarrow -a_n = |a_n| > 0 \quad \text{" "}$$

$$\Rightarrow a_n < 0 \quad \text{" "}$$

better to do directly at 1st case

unless all $a_n > 0$, which implies a contradiction of the 1st case.

Ex. 2.3.3 (Sq. Thm.)

Consider three sequences (x_n), (y_n), (z_n) satisfying

(1) $x_n \le y_n \le z_n$

(2) $\underbrace{\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n}_{x_n \text{ \& } z_n \text{ are convergent with the same limit } L} = L$

Then, $y_n \to L$ converges, too, & also to L.

pf: $x_n \to L$ & $z_n \to L \implies \forall \varepsilon > 0$,

$\exists N_1, N_2 \in \mathbb{N}$ s.t. $n \ge N := \max\{N_1, N_2\}$

$\implies |x_n - L| < \varepsilon$ & $|z_n - L| < \varepsilon$

$\implies$ $\boxed{\begin{array}{l} -\varepsilon < x_n - L < \varepsilon \\ -\varepsilon < z_n - L < \varepsilon \end{array}}$ (*)

Now,

$$x_n \le y_n \le z_n \quad -L \qquad (**)$$

$$\Longrightarrow \quad \underbrace{-\varepsilon \overset{(*)}{<} x_n - L \overset{(**)}{\le} y_n - L \overset{(**)}{\le} z_n - L \overset{(*)}{<} \varepsilon}$$

$$\Longrightarrow \quad -\varepsilon < y_n - L < \varepsilon$$

$$\Longrightarrow \quad |y_n - L| < \varepsilon$$

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Thm. (Examples)

(1) If $p>0$, then $\boxed{\lim_{n\to\infty} \frac{1}{n^p} = 0.}$

(2) If $p>0$, then $\boxed{\lim_{n\to\infty} p^{1/n} = 1.}$ $\left(= p^0 = p^{\lim_{n\to\infty} 1/n}\right)$

(3) If $p>0$, $a\in\mathbb{R}$, then

$$\boxed{\lim_{n\to\infty} \frac{n^a}{(1+p)^n} = 0}$$

e.g. $\frac{\sqrt{n}}{(1+1/2)^n} = \frac{\sqrt{n}}{(\frac{3}{2})^n} \to 0$

(4) $\boxed{\lim_{n\to\infty} n^{1/n} = 1}$

(5) $\boxed{\lim_{n\to\infty} x^n = 0}$ for all $\boxed{-1<x<1}$ (i.e. $\boxed{|x|<1}$)

(6) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ exists for some seq. $(a_n)_{n\in\mathbb{N}}$,

then: (a) $\boxed{L<1 \implies \lim_{n\to\infty} a_n = 0}$

(b) $\boxed{L>1 \implies \lim_{n\to\infty} |a_n| = \infty \text{ or } \lim_{n\to\infty} a_n = \pm\infty}$

pf: (1) Let $\varepsilon > 0$ be given. By the A.P. there is an $N \in \mathbb{N}$ s.t.

$$\frac{1}{N} < \varepsilon^{1/p}$$

so

$$n \geq N \implies \frac{1}{n} \leq \frac{1}{N} < \varepsilon^{1/p}$$

$$\implies \frac{1}{n^p} < \varepsilon$$

$$\implies \left|\frac{1}{n^p} - 0\right| = \frac{1}{n^p} < \varepsilon \quad \checkmark$$

(2) Let $\varepsilon > 0$, & consider the cases

(i) $p = 1$

(ii) $p > 1$

(iii) $p < 1$

(i) If $p = 1$, then $p^{1/n} = 1$ for all $n \in \mathbb{N}$, so clearly $|p^{1/n} - 1| = 0 < \varepsilon$, & so $p^{1/n} \to 1$.

(ii) If $p > 1$, let $x_n := p^{1/n} - 1$ & use:

$$1 + n x_n \le 1 + x_n + \binom{n}{2} x_n^2 + \binom{n}{3} x_n^3 + \cdots + \binom{n}{n-1} x_n^{n-1} + x_n^n$$

$$= (1 + x_n)^n \quad \leftarrow \text{Binomial thm.}$$

$$= (\cancel{1} + p^{1/n} - \cancel{1})^n$$

$$= p$$

$$\Longrightarrow 0 < n x_n \le p - 1$$

$$\Longrightarrow \quad 0 < x_n < \frac{p-1}{n} \longrightarrow 0 \text{ at } n \to \infty \text{ (by (1))}$$

hence does ~~not~~ apply: 0 by Order Limit Law (2) only if we already knew $x_n \to x$ were convergent.

Either use the Squeeze thm. now (Ex. 2.3.3),

or find $N \in \mathbb{N}$ using A.P. s.t.

$$\frac{1}{N} < \frac{\varepsilon}{(p-1)}$$

$$\Longrightarrow \text{ if } n \ge N, \quad \frac{1}{n} \le \frac{1}{N} < \frac{\varepsilon}{(p-1)}$$

$\Longrightarrow$

$$\Longrightarrow \quad |x_n - 0| = x_n < \frac{p-1}{n}$$

$$< \cancel{(p-1)} \cdot \frac{\varepsilon}{\cancel{p-1}}$$

$$= \varepsilon$$

$$\Longrightarrow \lim_{n\to\infty} x_n = \underbrace{\lim_{n\to\infty} (p^{1/n} - 1) = 0}$$

$$+$$

$$\Longrightarrow \lim_{n\to\infty} p^{1/n} = 1 \text{ since } \overbrace{\lim_{n\to\infty} 1 = 1}$$

$$\Longrightarrow \lim_{n\to\infty} (p^{1/n} - 1) + \lim_{n\to\infty} 1$$

$$= 0 + 1$$

$$= 1$$

$$\Longrightarrow \lim_{n\to\infty} p^{1/n} = \lim_{n\to\infty} (p^{1/n} - 1 + 1)$$

$$= 0 + 1 = 1$$

(iii) If $0 < p < 1$, then $q = \frac{1}{p} > 1 \overset{(ii)}{\Longrightarrow}$

$$1 = \lim_{n\to\infty} q^{1/n} = \lim_{n\to\infty} \frac{1}{p^{1/n}}$$

$$\Longrightarrow 1 = \lim_{n\to\infty} p^{1/n}.$$

(4) Define $x_n := n^{1/n} - 1$

≥ 0

$$\Longrightarrow \quad \binom{n}{2} x_n^2 \leq \sum_{k=0}^{n} \binom{n}{k} x_n^k$$

$$= (1+x_n)^n$$

$$= (\cancel{1} + \overbrace{n^{1/n} \cancel{-1}})^n$$

$$= n$$

i.e.

$$\frac{\cancel{n}(n-1)}{2} x_n^2 \leq \cancel{n}$$

$$\Longrightarrow \quad x_n^2 \leq \frac{2}{n-1}$$

$$\Longrightarrow \quad 0 \leq x_n \leq \frac{\sqrt{2}}{\sqrt{n-1}} \qquad (*)$$

$$= \sqrt{2} \cdot \frac{1}{\underbrace{(n-1)}_{=m}^{\overbrace{1/2}^{=p}}} = \sqrt{2} \cdot \frac{1}{m^p}$$

now apply (1)

$$\downarrow$$

$$\sqrt{2} \cdot 0 = 0$$

$\Longrightarrow$ $x_n \to 0$ by Squeeze thm, or else use any $\varepsilon > 0$ on RHS of (*).

(3) <u>If $a \le 0$,</u> then $\frac{n^a}{(1+p)^n} = \frac{1}{(1+p)^n} \cdot \frac{1}{n^{-a}}$

<u>Case I</u>

$$\frac{1}{(1+p)^n} \to 0 \qquad \frac{1}{n^{-a}} \to 0 \text{ by (1) if } a<0 \quad \text{or} \quad \to 1 \text{ if } a=0$$

That $\frac{1}{(1+p)^n} \to 0$ is shown directly: Let $\varepsilon > 0$, find $N \in \mathbb{N}$ using A.P. s.t.

$$\frac{1}{N} < \frac{\ln(1+p)}{\ln(1/\varepsilon)}$$

Then,

$$n \ge N \Rightarrow \frac{1}{n} \le \frac{1}{N} < \frac{\ln(1+p)}{\ln(1/\varepsilon)}$$

$$\Rightarrow \ln(1/\varepsilon) < n \ln(1+p) = \ln((1+p)^n)$$

$$\Rightarrow \frac{1}{\varepsilon} < (1+p)^n$$

$$\Rightarrow \frac{1}{(1+p)^n} < \varepsilon$$

Case 2 If $a>0$, use A.P. twice:

- to find $k>a$, $k\in\mathbb{N}$
- to find $N>2k$, $N\in\mathbb{N}$

Then $n\geq N>2k$, too, &

$$n>2k$$

$$\Rightarrow \frac{n}{2}>k\geq k-l \quad (1\leq l\leq k)$$

$$\Rightarrow \frac{n}{2}-k+l\geq 0$$

$+n/2$ $\quad$ $+n/2$

$$\Rightarrow \boxed{n-k+l\geq\frac{n}{2}, \text{ for all } 1\leq l\leq k}$$

Now use this + Binomial Thm:

$$\frac{n^k p^k}{2^k\cdot k!}=\underbrace{\frac{n}{2}\cdots\frac{n}{2}}_{k\text{ times}}\cdot\frac{p^k}{k!}$$

$$\leq\frac{n(n-1)\cdots(n-k+1)}{k!}\cdot p^k$$

$$=\binom{n}{k}p^k$$

$$\leq\sum_{k=0}^{n}\binom{n}{k}p^k=(1+p)^n$$

Thus,

$$\frac{n^k p^k}{2^k k!} \le (1+p)^n$$

$$\cdot\, n^a$$

$$\Longrightarrow \quad \frac{1}{(p+1)^n} \le \frac{2^k k!}{p^k} \cdot \frac{1}{n^k}$$

$$\Longrightarrow \quad 0 \le \frac{n^a}{(p+1)^n} \le \underbrace{\underbrace{\frac{2^k k!}{p^k}}_{\text{const w.r.t. } n} \cdot \underbrace{\frac{1}{n^{k-a}}}_{= \frac{1}{n^p},\ p = k-a \ \to\ 0 \text{ by (1)}}}_{\text{RHS} \to 0}$$

(since $k > a$, $k - a > 0$)

By Sq. Thm.,

$$\frac{n^a}{(p+1)^n} \longrightarrow 0$$

(5) Let $|x| < 1$ be given, & define

$$p := \frac{1}{|x|} - 1 > 0$$

$$a = 0$$

$$\Longrightarrow \frac{n^a}{(1+p)^n} = \frac{n^0}{\left(1 + \frac{1}{|x|} - 1\right)^n} = |x|^n$$

$\downarrow$

0 by (4)

(6) (a) $L < 1 \Longrightarrow \left|\frac{x_{n+1}}{x_n}\right| < 1$ for some $n \in \mathbb{N}$

prelim. Prop

$\Longrightarrow \left|\frac{x_{n+1}}{x_n}\right| < r$ for some $r \in \mathbb{R}$, $r < 1$

(Density of $\mathbb{Q}$ in $\mathbb{R}$, Thm. 1.4.3)

$\Longrightarrow \boxed{|x_{n+1}| < r|x_n| < |x_n|}$

(since $r < 1$)

$\Longrightarrow x_n$ is decreasing

Iterate $|x_{n+1}| < r|x_n|$: fix $N \in \mathbb{N}$ & observe
(so $\left|\frac{x_{N+1}}{x_N}\right| < 1$)

$$0 < |x_{N+k}| < r|x_{N+k-1}| < \cdots < r^k|x_N|$$

$|x_{N+k}| \to 0$ as $k \to \infty$ by Squeeze Thm. $\qquad r^k|x_N| \to 0$

(b) $L > 1$: <u>Exercise!</u>

Thm. 2.4.2 (Monotone Convergence Thm.)

Any bounded monotone (incr./decr.) sequence is convergent.

pf: WOLOG supp. $(a_n)_{n\in\mathbb{N}}$ increasing & bdd:

(i) $a_n \leq a_{n+1}$

(ii) $|a_n| \leq M$ for all $n\in\mathbb{N}$, for some $M>0$

Let

$$\boxed{s := \sup\{a_n \mid n\in\mathbb{N}\}}$$

and let us prove

$$\boxed{\lim_{n\to\infty} a_n = s}$$

Let $\varepsilon>0$, & use Lemma 1.3.8 to find a_N s.t.

$$\underbrace{s-\varepsilon < a_N}_{\text{Lemma 1.3.8}} \underbrace{\leq a_n \text{ for all } n \geq N}_{\text{Pr. (i)}}$$

$\Longrightarrow \quad \forall n \geq N,$

$$s-\varepsilon < a_n \leq s < s+\varepsilon$$

$\Longrightarrow \quad -\varepsilon < a_n - s < \varepsilon$

$\Longrightarrow \quad |a_n - s| < \varepsilon$ <u>QED</u>

<u>ex.</u> (Ex. 2.4.3 (a)) Define the <u>infinite surd</u>

$$a_1 = \sqrt{2}$$

$$a_2 = \sqrt{2 + a_1} = \sqrt{2+\sqrt{2}}$$

$$a_3 = \sqrt{2 + a_2} = \sqrt{2+\sqrt{2+\sqrt{2}}}$$

$$\vdots$$

In gen.,

$$a_1 = \sqrt{2}$$

$$a_{n+1} = \sqrt{2 + a_n}$$

Now,

(i) $a_2 = \sqrt{2 + a_1} \geq \sqrt{2} = a_1$, while

$$a_n \leq a_{n+1} \Rightarrow a_{n+1} = \sqrt{2 + a_n}$$
$$\leq \sqrt{2 + a_{n+1}}$$
$$= a_{n+2}$$

So by induction, $a_n \leq a_{n+1}$ $\forall n \in \mathbb{N}$.

(ii) Observe, now, that $M = \sqrt{2} + 1$ is an upper bound, since

$$a_1 = \sqrt{2} \leq \sqrt{2} + 1 = M$$

and if

$$a_n \leq M$$

then

$$a_{n+1} = \sqrt{2 + a_n} \leq \sqrt{2 + M}$$
$$= \sqrt{2 + \sqrt{2} + 1}$$
$$\leq \sqrt{2 + 2\sqrt{2} + 1}$$
$$= \sqrt{(\sqrt{2}+1)^2} = \sqrt{2} + 1 = M$$

So by induction $a_n \leq M$ $\forall n \in \mathbb{N}$.

Thus $\lim_{n\to\infty} a_n = a$ exists by the M.C.T.

Since

$$a_{n+1} = \sqrt{2+a_n}$$

&

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = a$$

we have

$$a = \lim_{n\to\infty} a_{n+1}$$

$$= \lim_{n\to\infty} \sqrt{2+a_n}$$

$$= \sqrt{2+a}$$

$$\Longrightarrow \quad a^2 = 2+a$$

$$\Longrightarrow \quad a^2 - a - 2 = 0$$

only +

$$\Longrightarrow \quad a = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2}$$

$$= 2$$