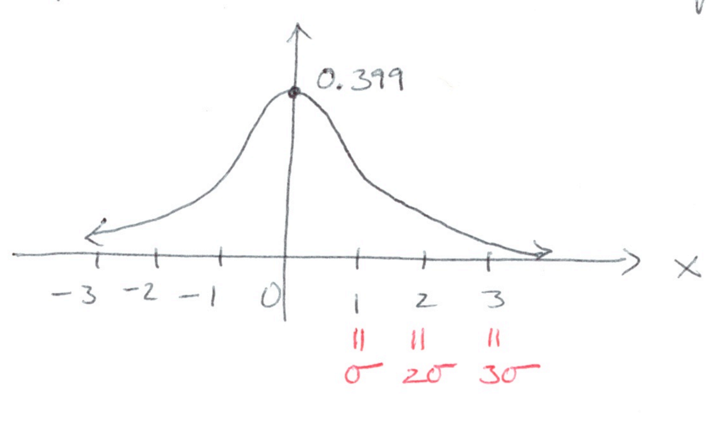
Normal Random Variables

We say X is a <u>normal random variable</u>, or that X is <u>normally distributed</u>, <u>with parameters</u> μ & σ^2, if

$$\boxed{\begin{array}{l} f : R(X) = \mathbb{R} \longrightarrow \mathbb{R} \\ \text{is given by} \\ f(x) = \dfrac{1}{\sqrt{2\pi}\cdot\sigma}\, e^{-(x-\mu)^2/2\sigma^2} \end{array}}$$

<u>ex.</u> $\mu = 0$ & $\sigma = 1$, $f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$



Prop. : If X is normally distributed with parameters μ & σ^2, then

$$\boxed{Y \overset{\text{def}}{=} aX + b}$$

is normally distributed with parameters $a\mu + b$ and $a^2\sigma^2$.

Pf: Suppose $a > 0$ first. Then the cdf of Y is

$$F_Y(x) = P(Y \le x)$$
$$= P(aX + b \le x)$$
$$= P\left(X \le \frac{x-b}{a}\right)$$
$$= F_X\left(\frac{x-b}{a}\right)$$

$$\Longrightarrow f_Y(x) = \frac{d}{dx} F_Y(x) = \frac{d}{dx} F_X\left(\frac{x-b}{a}\right)$$
$$= f_X\left(\frac{x-b}{a}\right) \cdot \frac{1}{a}$$
$$= \frac{1}{\sqrt{2\pi}\, a\sigma} e^{-\left(\frac{x-b}{a} - \mu\right)^2 / 2\sigma^2}$$

$$= \frac{1}{\sqrt{2\pi} \cdot \underbrace{a\sigma}_{a\sigma}} e^{-(x - \underbrace{b - a\mu}_{b+a\mu})^2 / 2\underbrace{(a\sigma)}_{a\sigma}^2}$$

$\boxtimes$

Remark: Let us apply this proposition to the special case of

$$\boxed{Z \stackrel{\text{def}}{=} \frac{X-\mu}{\sigma} = \frac{1}{\sigma} X - \frac{\mu}{\sigma}}$$

Then, Z is normally distributed with parameters

$$\mu_Z = \frac{1}{\sigma}\mu - \frac{\mu}{\sigma} = 0$$

$$(a\mu + b)$$

$$\sigma_Z = \left(\frac{1}{\sigma}\right)^2 \sigma^2 = 1$$

$$(a^2\sigma^2)$$

which is nothing but the standard/unit normal distribution.

Prop. 1 If X is normally distributed with parameters μ and σ^2, then

$$\boxed{\begin{array}{ll} (1) & \displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1 \\ (2) & E[X] = \mu \\ (3) & Var(X) = \sigma^2 \end{array}}$$

Pf: (1) Let $y = \frac{x-\sigma}{\mu}$ so that we can write $f(x)$ as $f(y) = e^{-y^2/2}$. Then,

$$I \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2} dx$$

$$\left(\begin{array}{l} y = \frac{x-\mu}{\sigma} \\ dy = \frac{1}{\sigma} dx \\ \sigma\, dy = dx \end{array} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy$$

$$\Longrightarrow I^2 = \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} e^{-y^2/2} dy \right)^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-y^2/2} dy \int_{-\infty}^{\infty} e^{-x^2/2} dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} dy\,dx$$

We change to polar coordinates:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

The Jacobian of this change is

$$\det\begin{pmatrix} \partial_r x & \partial_\theta x \\ \partial_r y & \partial_\theta y \end{pmatrix} = \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r$$

So

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\,dy\,dx$$

$$= \frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi} e^{-r^2/2}\cdot r\,d\theta\,dr$$

$$= \frac{2\pi}{2\pi}\int_0^{\infty} e^{-r^2/2}\,r\,dr \qquad \begin{pmatrix} u = r^2/2 \\ du = r\,dr \end{pmatrix}$$

$$= \int_0^{\infty} e^{-u}\,du$$

$$= \lim_{b\to\infty}\left[-e^{-u}\right]_0^b = 1 - \lim_{b\to\infty} e^{-b} = 1$$

$Z = \frac{X-\mu}{\sigma}$ (the standard normal distribution with $\mu_Z = 0$, $\sigma_Z = 1$)

(2) $$E[Z] = \int_{-\infty}^{\infty} x f_Z(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-x^2/2}\,dx \qquad \left(\begin{array}{l} u = \frac{x^2}{2} \\ du = x\,dx \end{array} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u}\,du$$

$$= -\frac{1}{\sqrt{2\pi}} \left[e^{-u} \right]_{+\infty}^{\infty}$$

$$= 0$$

$$\Longrightarrow \quad 0 = E[Z] = E\left[\frac{1}{\sigma}X - \frac{\mu}{\sigma}\right]$$

$$= \frac{1}{\sigma} E[X] - \frac{\mu}{\sigma}$$

$$\Longrightarrow \quad E[X] = \mu$$

(3) $$Var(Z) = E[Z^2] - \overbrace{E[Z]}^{=0}{}^2$$

$$= E[Z^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx$$

$$\left(\begin{array}{ll} u = x & dv = x e^{-x^2/2}\,dx \\ du = dx & v = -e^{-x^2/2} \end{array} \right)$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} \left[x e^{-x^2} \right]_{-\infty}^{\infty}}_{=0} + \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\,dx}_{\sqrt{2\pi}} = 1$$

$$\Rightarrow 1 = Var(Z) = Var\left(\tfrac{1}{\sigma}X - \tfrac{\mu}{\sigma}\right)$$

$$= \frac{1}{\sigma^2} Var(X)$$

Prop. 3 in the general theory notes.

$$\Rightarrow \quad Var(X) = \sigma^2 .$$

<u>QED</u>

<u>Notation:</u> The <u>customary notation</u> for the <u>cdf</u> is not F or F_X, but Φ or Φ_X, (Greek "F")

$$\boxed{\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt}$$

$$= P(X \leq x)$$

$\boxtimes$

<u>Observation:</u> For the standard normal distribution Z we have

$$\cancel{\Phi(-x) = P(Z \leq -x)}$$

$$\cancel{= P(-Z \geq x)}$$

$$\cancel{= 1 - P(-Z \leq x)}$$

$$\cancel{= 1 - P(Z \geq x)}$$

$$=$$

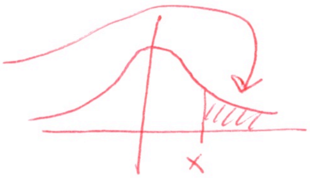
$$\boxed{\Phi(-x) = 1 - \Phi(x)}$$

pf: $\Phi(-x) = P(Z \le -x)$

* $e^{-x^2/2}$ is symmetric:

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-x} e^{-t^2/2}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-t^2/2}\, dt$$



$$= \underbrace{1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt}_{= 1 - P(Z \le x)}$$

$$= 1 - \Phi(x)$$

QED

Corollary: For any other random variable X, we recall from the proof of Prop. 1 that $Z = \frac{X-\mu}{\sigma}$ is the standard normal distribution. Hence,

$$\boxed{\begin{aligned} F_X(a) = P(X \le a) &= P\Big(\underbrace{\tfrac{X-\mu}{\sigma}}_{=Z} \le \tfrac{a-\mu}{\sigma}\Big) \\ &= \Phi\Big(\frac{a-\mu}{\sigma}\Big) \end{aligned}}$$

ex. 1 Suppose X is a normal random variable with parameters $\mu = 3$, $\sigma^2 = 9$. Find

(a) $P(2 < X < 5)$

(b) $P(X > 0)$

(c) $P(|X-3| > 6)$

A: (a) $P(2 < X < 5) = P\left(\underbrace{\frac{2-3}{3}}_{-1/3=} < \underbrace{\frac{X-3}{3}}_{Z=} < \underbrace{\frac{5-3}{3}}_{2/3=}\right)$

$$= P(-1/3 < Z < 2/3)$$

$$= \Phi(2/3) - \Phi(-1/3)$$

$$= \Phi(2/3) - (1 - \Phi(1/3))$$

$$\boxed{\begin{aligned} &= \Phi(2/3) + \Phi(1/3) - 1 \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{2/3} e^{-t^2/2}\,dt + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{1/3} e^{-t^2/2}\,dt - 1 \\ &\approx 0.3779 \end{aligned}}$$

(b) $P(X > 0) = P\left(\frac{X-3}{3} > \frac{-3}{3}\right)$

$$= P(Z > -1)$$

$$= 1 - P(Z \leq -1)$$

$$= 1 - \Phi(-1)$$

$$= 1 - (1 - \Phi(1))$$

$$\boxed{\begin{aligned} &= \Phi(1) \\ &\approx 0.8413 \end{aligned}}$$

(c) $P(|X-3| > 6)$

$$= 1 - P(|X-3| \leq 6)$$

$$= 1 - P(-6 \leq X-3 \leq 6)$$

$$= 1 - P(-3 \leq X \leq 9)$$

$$= 1 - [P(X \leq 9) - P(X \geq -3)]$$

$$= \cancel{1} - [P(X \leq 9) - \cancel{1} + P(X \leq -3)]$$

no cancellation! oops

(b) $P(X>0) = P\left(\frac{X-3}{3} > \frac{0-3}{3}\right)$

$= P(Z > -1)$

$= 1 - P(Z \le -1)$

$= 1 - \Phi(-1)$

$= 1 - (1 - \Phi(1))$

$\boxed{= \Phi(1) \approx 0.8413}$

(c) $\underbrace{P(|X-3| > 6)}_{= P(\{X-3>6\} \sqcup \{-(X-3)>6\})} = P(X-3>6) + P(-(X-3)>6)$

$= P(X>9) + P(X<-3)$

$= 1 - P(X \le 9) + P(X<-3)$

$= 1 - P\left(\frac{X-3}{3} \le \frac{9-3}{3}\right) + P\left(\frac{X-3}{3} < \frac{-3-3}{3}\right)$

$= 1 - P(Z \le 2) + P(Z < -2)$

$= 1 - \Phi(2) + \Phi(-2)$

$= 1 - \Phi(2) + 1 - \Phi(2)$

$\boxed{= 2 - 2\Phi(2) \approx 0.0456}$

Recall the discrete binomial random variable, which we will here denote

$$S_n = \# \Sigma_i \text{ in a sequence } (\tau_i)_{i=1}^{n}$$
of Bernoulli trials, $\tau_i \in \{\Sigma, F\}$
with $p = P(\Sigma)$

Its expected value & variance are

$$\mu = E[S_n] = np$$

$$\sigma^2 = Var(S_n) = np(1-p)$$

<u>Thm. (DeMoivre–Laplace)</u> For all $a < b$,

$$\boxed{\lim_{n\to\infty} P\Big(a \le \underbrace{\frac{S_n - np}{\sqrt{np(1-p)}}}_{= \frac{S_n-\mu}{\sigma} \text{ "} = Z\text{"}} \le b\Big) = \Phi(b) - \Phi(a)}$$

Equivalently,

$$\lim_{n\to\infty} \frac{S_n - np}{\sqrt{np(1-p)}} = Z$$

standard normal random variable

or, since $P(S_n = k) = \binom{n}{k} p^k (1-p)^{n-k}$,

$$\lim_{n\to\infty} \frac{P(S_n = k) = \binom{n}{k} p^k (1-p)^{n-k}}{\frac{1}{\sqrt{2\pi}\sqrt{np(1-p)}} \cdot e^{-(k-np)^2/2np(1-p)}} = 1$$

pf: This is a special case of the central limit thm. (Ch. 8), so let's wait!