Poisson Random Variable

A Poisson random variable with parameter $\lambda > 0$, is defined in terms of its probability mass function. It is a discrete random variable whose pmf is given by

$$\boxed{p(i) = P(X=i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}}$$

$(i = 0, 1, \dots \text{ i.e. } i \in \mathbb{N}_0)$

That is,

$$\boxed{\begin{array}{l} X : S \to \mathbb{R} \\ R(X) = \mathbb{N}_0 \end{array}}$$

Prop. 1 A Poisson random variable satisfies:

$$\boxed{\begin{array}{ll} (1) & \sum_{i=0}^{\infty} p(i) = 1 \\ (2) & E[X] = \lambda \\ (3) & Var(X) = \lambda \end{array}}$$

pf: (1) $\sum_{i=0}^{\infty} p(i) = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!}$

$$= e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}$$

$$= e^{-\lambda} e^{\lambda}$$

$$= 1$$

(2) $E[X] = \sum_{i=0}^{\infty} i p(i)$

$$= \sum_{i=1}^{\infty} \cancel{i} \cdot e^{-\lambda} \cdot \frac{\lambda^i}{\cancel{i!}\,(i-1)!}$$

$= e^{-\lambda}\lambda \sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!}$ then re-index

$$= e^{-\lambda} \cdot \lambda \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}$$

$$= e^{-\lambda} \cdot \lambda \cdot e^{\lambda}$$

$$= \lambda$$

(3) $\mathrm{Var}(X) = E[X^2] - E[X]^2$

$$= \sum_{i=0}^{\infty} i^2 p(i) - \lambda^2$$

$$= \sum_{i=1}^{\infty} i^{\cancel{2}} \cdot e^{-\lambda} \cdot \frac{\lambda^i}{\cancel{i!}\,(i-1)!} - \lambda^2$$

$$= \lambda \sum_{i=1}^{\infty} \frac{i e^{-\lambda} \lambda^{i-1}}{(i-1)!} - \lambda^2$$

$$\left(\begin{array}{c} j = i-1 \\ \Leftrightarrow i = j+1 \end{array} \right) \qquad = \lambda \sum_{j=0}^{\infty} \frac{(j+1) e^{-\lambda} \lambda^{j}}{j!} - \lambda^2$$

$$= \lambda \sum_{j=0}^{\infty} \frac{\cancel{j} e^{-\lambda} \lambda^{j}}{\cancel{j!}\ (j-1)!} + \lambda \sum_{j=0}^{\infty} \frac{e^{-\lambda} \lambda^{j}}{j!} - \lambda^2$$

$$= \lambda^2 e^{-\lambda} \underbrace{\sum_{j=1}^{\infty} \frac{\lambda^{j-1}}{(j-1)!}}_{= e^{\lambda}} + \lambda e^{-\lambda} \underbrace{\sum_{j=0}^{\infty} \frac{\lambda^{j}}{j!}}_{= e^{\lambda}} - \lambda^2$$

$$= \lambda^2 e^{-\lambda} e^{\lambda} + \lambda e^{-\lambda} e^{\lambda} - \lambda^2$$

$$= \cancel{\lambda^2} + \lambda - \cancel{\lambda^2}$$

$$= \lambda$$

QED

The Poisson random variable may be used as an approximation to a binomial distribution with parameters (n, p) when

(1) n is large
(2) p is small
} so that np is of moderate size

For then, letting $\boxed{\lambda = np,}$ if X is binomial,

$$P(X=i) = \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \frac{n(n-1)\cdots(n-i+1)}{i!} \cdot \left(\frac{\lambda}{n}\right)^i \cdot \left(1 - \frac{\lambda}{n}\right)^{n-i} \qquad \left(p = \frac{\lambda}{n}\right)$$

$$= \overbrace{\frac{n(n-1)\cdots(n-i+1)}{n^i}}^{\approx 1} \cdot \frac{\lambda^i}{i!} \cdot \frac{\overbrace{(1-\lambda/n)^n}^{\approx e^{-\lambda}}}{\underbrace{(1-\lambda/n)^i}_{\approx 1}}$$

$$\approx \frac{\lambda^i}{i!} e^{-\lambda}$$

Prop. 2 (Recursion formula for computing Poisson distribution function)

$$\boxed{\begin{aligned} P(X=i+1) &= \frac{\lambda}{i+1} P(X=i) \\ &= \frac{\lambda^{i+1}}{(i+1)!} \overbrace{P(X=0)}^{=e^{-\lambda}} \end{aligned}}$$

Therefore,

$$P(X=1) = \lambda P(X=0)$$

$$P(X=2) = \frac{\lambda}{2} P(X=1) = \frac{\lambda^2}{2} P(X=0)$$

$$P(X=3) = \frac{\lambda}{3} P(X=2) = \frac{\lambda^3}{3!} P(X=0)$$

$$\vdots$$

so

$$\boxed{\begin{aligned} F(k) &= \sum_{i=0}^{k} P(X=i) \\ &= \left(\sum_{i=0}^{k} \frac{\lambda^i}{i!}\right) \underbrace{P(X=0)}_{=e^{-\lambda}} \\ &= e^{-\lambda} \sum_{i=0}^{k} \frac{\lambda^i}{i!} \end{aligned}}$$

pf:

$$\frac{P(X=i+1)}{P(X=i)} = \frac{\cancel{e^{-\lambda}} \cdot \frac{\lambda^{\cancel{i+1}}}{(i+1)\cancel{!}}}{\cancel{e^{-\lambda}} \cdot \frac{\cancel{\lambda^{i}}}{\cancel{i!}}} = \frac{\lambda}{i+1}.$$

QED