Bernoulli Random Variables

&

Binomial

We consider here a special type of experiment, namely one with only two outcomes,

$$\boxed{\begin{aligned} S &= \{\text{success}, \text{failure}\} \\ &= \{\Sigma, F\} \end{aligned}}$$

(sample space)

(for short)

We suppose, further, that

$$\boxed{\begin{aligned} P(\Sigma) &= p \\ P(F) &= 1-p \end{aligned}}$$

This allows us to define the Bernoulli random variable

$$\boxed{\begin{aligned} &X : S \longrightarrow \mathbb{R} \\ &X(\Sigma) = 1 \\ &X(F) = 0 \end{aligned}}$$

whose probability mass function is

$$\boxed{\begin{aligned} &p : \mathbb{R} \to [0,1] \\ &p(0) = P(X=0) = P(F) = 1-p \\ &p(1) = P(X=1) = P(E) = p \end{aligned}}$$

Of course its expectation & variance are

$$\boxed{E[X] = 0 \cdot p(0) + 1 \cdot p(1) = p}$$

$$\boxed{\begin{aligned} Var(X) = E[(X-p)^2] \\ &= (0-p)^2 \cdot p(0) + (1-p)^2 \cdot p(1) \\ &= p^2(1-p) + (1-p)^2 p \\ &= p(1-p)(\cancel{p} + 1 - \cancel{p}) \\ &= p(1-p) \end{aligned}}$$

but the real interest is not in this type of experiment by itself, but iterated, repeated trials of it.

Let us now consider n repeated trials of a binary $\{\Sigma, F\}$-type experiment with $P(\Sigma) = p$, $P(F) = 1-p$: After n trials, the sample space consists of sequences $\tau_1\tau_2\cdots\tau_n$:

$$\boxed{S = \{\tau_1\tau_2\cdots\tau_n \mid \tau_i = \text{trial } \# i \in \{\Sigma_i, F_i\}\}}$$

<u>Ex</u> Flip a coin twice, & suppose $P(H) = p$, $P(T) = 1-p$. Then $n = 2$, and

$$S = \{TT, TH, HT, HH\}$$

where $\tau_1\tau_2 \in S$ could be any of the 4.

Then define the binomial random variable with parameters (n, p) by

$$\boxed{\begin{aligned} &X : S \to \mathbb{R} \\ &X(\tau_1\tau_2\cdots\tau_n) = \#\, \tau_i = \Sigma_i \end{aligned}}$$

Prop. 1

$$\left.\begin{aligned}\sum_{i=0}^{\infty} p(i) &= \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} \\ &= (p+(1-p))^n\end{aligned}\right\} \text{Binomial Thm.}$$

$$= 1^n$$

$$= 1$$

Prop. 2 $\boxed{E[X] = np}$

$$E[X^k] = np\, E[(Y+1)^{k-1}]$$

where Y has parameters $(n-1, p)$

$\boxed{Var(X) = np(1-p)}$ & $\boxed{\sigma = \sqrt{np(1-p)}}$

pf:

$$E[X] = \sum_{i=0}^{n} i\,p(i)$$

$$= \sum_{i=1}^{n} i\underbrace{\binom{n}{i}}_{=n\binom{n-1}{i-1} \text{ by } 1.10(d)} p^i (1-p)^{n-i}$$

factor out np

$$= np \sum_{i=1}^{n} \binom{n-1}{i-1} p^{i-1} (1-p)^{n-1-(i-1)}$$

Binom. Thm.

$$= np(\cancel{p} + 1 - \cancel{p})^{n-1} = np$$

Apply this result to $k=2$ so we can figure out $E[X^2]$:

$$E[X^2] = np\,E[Y+1]$$
$$= np\left(E[Y]+1\right)$$
$$= np\left((n-1)p+1\right)$$

Since

$$E[X] = np$$

we have

$$Var(X) = E[X^2] - E[X]^2$$
$$= np\left((n-1)p+1\right) - n^2p^2$$
$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$
$$= np(1-p)$$

QED

Prop. 3 Let X be a binomial random variable with parameters (n, p), where $0 < p < 1$. Then, $p(k) = P(X=k)$ increases monotonically, then decreases monotonically, reaching its max when $k = \max\{i \mid i \leq (n+1)p\}$.

pf: Observe:

$$\frac{p(k)}{p(k-1)} = \frac{P(X=k)}{P(X=k-1)}$$

$$= \frac{\binom{n}{k} p^k \cancel{(1-p)^{n-k}}}{\binom{n}{k-1} \cancel{p^{k-1}} \cancel{(1-p)^{n-k+1}}}$$

$$= \frac{\frac{n!}{k!(n-k)!} \cdot p}{\frac{n!}{(k-1)!(n-k+1)!} \cdot (1-p)}$$

$$= \frac{\cancel{n!}}{k\cancel{!}\cancel{(n-k)!}} \cdot \frac{\cancel{(k-1)!}(n-k+1)\cancel{!}}{\cancel{n!}} \cdot \frac{p}{1-p}$$

$$= \frac{(n-k+1)p}{k(1-p)}$$

Thus,

$$p(k) \geq p(k-1) \iff \frac{p(k)}{p(k-1)} \geq 1$$

$$\iff \frac{(n-k+1)}{k} \cdot \frac{p}{1-p} \geq 1$$

$$\iff (n-k+1)p \geq k(1-p)$$

$$\iff np - \cancel{kp} + p \geq k - \cancel{kp}$$

$$\iff (n+1)p \geq k$$

QED

Corollary 4: $\boxed{p(k+1) = \left(\frac{p}{1-p}\right)\left(\frac{n-k}{k+1}\right)p(k)}$

pf: Use $k+1$ & k instead of k & $k-1$ above.

Application: The binomial probability distribution function

$$F(k) = P(X \leq k)$$

$$= \sum_{i=0}^{k} P(X=k)$$

$$= \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i}$$

can be computed recursively using Cor. 4