7/10/20

Def A set A is said to be <u>countable</u> if it is in bijective correspondence with $\mathbb{N}$, i.e. if there exists a bijection

$$f : A \xrightarrow{\sim} \mathbb{N}$$

$$(\text{or } g : \mathbb{N} \xrightarrow{\sim} A)$$

$(\mathbb{N} = \{1, 2, \dots\}$ here$)$

$\boxtimes$

Prop 1 $\mathbb{Z}$ is countable.

pf: Define

$$f : \mathbb{N} \to \mathbb{Z}$$

by

$$f(n) = \begin{cases} \frac{n-1}{2}, & \text{if } n = 2k+1 \text{ is odd} \\ -\frac{n}{2}, & \text{if } n = 2k \text{ is even} \end{cases}$$

e.g. $f(1) = 0$, $f(2) = -1$

$f(3) = 1$ $\quad f(4) = -2$

$\vdots \qquad \vdots$

(odds $\to$ nonneg's
evens $\to$ negs.)

(ii) Secondly, f is <u>onto</u>: let $m \in \mathbb{Z}$ and let us find $n \in \mathbb{N}$ s.t. $f(n) = m$.

Case 1: $m \leq 0$.

In this case, let $n = -2m$ and observe: $f(n) = \frac{-n}{2} = \frac{-(-2m)}{2} = m$.

Case 2: $m > 0$.

In this case let $n = 2m+1$, & note

$$f(n) = \frac{n-1}{2} = \frac{2m+1-1}{2} = m$$

QED

(i) f is, first of all, one-to-one:

$f(n) = f(m) \implies n, m$ both odd or both even

(pf: $n = 2k$, $m = 2l+1$, say.

$\implies f(n) = -\frac{2k}{2} = -k$

$f(m) = \frac{2l+1-1}{2} = l$

$\implies -k = l$, but

$l, k \in \mathbb{N}$, so $l, k \geq 1$

& l cannot be ≥ 1 & < 1)

$\implies$ case 1: $n = 2k$, $m = 2l$:

$f(n) = f(m)$

$\implies -\frac{2k}{2} = -\frac{2l}{2}$

$\implies k = l$

$\implies m = n$

case 2: $n = 2k+1$, $2l+1$:

$f(n) = f(m) \implies$

$\frac{2k+1-1}{2} = \frac{2l+1-1}{2}$

$\implies k = l \implies m = n.$

<u>Thm. 1.5.6</u> (i) $\mathbb{Q}$ is countable.

(ii) $\mathbb{R}$ is uncountable.

<u>pf:</u> (i) Define

$$A_1 := \{0\}$$

and for $n \geq 2$

$$A_n := \{\pm p/q \mid p, q \in \mathbb{N},\ p/q \text{ is fully reduced, and } p+q = n\}$$

e.g. $A_2 = \{\frac{1}{1}, \frac{-1}{1}\}$

$A_3 = \{\frac{1}{2}, -\frac{1}{2}, \frac{2}{1}, -\frac{2}{1}\}$

$\vdots$

and define

$$f : \mathbb{N} \to \mathbb{Q}$$

by enumerating the elements in each A_n, as follows:

$$
\begin{aligned}
A_1 &\left\{\; f(1) = 0 \right. && (\text{the only element of } A_1 = \{0\}) \\
A_2 &\left\{ \begin{array}{l} f(2) = 1 \\ f(3) = -1 \end{array} \right. && (= \tfrac{1}{1}, \text{ the 1st element of } A_2 = \{1, -1\}) \\
& && \text{etc.} \\
& && \vdots \\
A_3 &\left\{ \begin{array}{l} f(4) = \tfrac{1}{2} \\ f(5) = -\tfrac{1}{2} \\ f(6) = 2 \\ f(7) = -2 \end{array} \right. \\
&\quad \vdots
\end{aligned}
$$

Since $A_n \cap A_m = \emptyset$ if $n \neq m$ and $\bigcup_{n=1}^{\infty} A_n = \mathbb{Z}$

(bec $p+q = n \neq m$)

(bec. each $p/q \in \mathbb{Q}$ occurs in exactly one A_n, the one for which $n = p+q$)

we are guaranteed injectivity (1-1) & surjectivity (onto) for f.

(II) A proof via Nested Interval Thm:

Suppose there is a bijection $f: \mathbb{N} \xrightarrow{\sim} \mathbb{R}$, & list the real #s:

$$f(1) = x_1$$
$$f(2) = x_2$$
$$\vdots$$

(x_i's are distinct bec. f is 1-1)

Let I_1 be a closed interval not containing x_1 (e.g. since $\exists n \in \mathbb{N}$ s.t. $x_1 < n$, we can take $I_1 = [n, n+1]$, for ex.)

$I_2 \subseteq I_1$, s.t. $x_2 \notin I_2$ (if $x_2 \notin I_1$, don't worry, let $I_2 = I_1$, while if $x_2 \in I_1$, then say $n \leq x_2 \leq n+1$, so either $x_2 \in [n, \frac{2n+1}{2}]$ or $x_2 \in (\frac{2n+1}{2}, n+1]$, so just pick I_2 in the half not containing x_2). In gen.:

$$\boxed{\begin{array}{l} \bullet\ I_{n+1} \subseteq I_n \\ \bullet\ x_{n+1} \notin I_{n+1} \end{array}}$$

Then no x_i lies in $\bigcap_{n=1}^{\infty} I_n$ bec. $x_i \notin I_i$,

& since $\mathbb{R} = \{x_i \mid i \in \mathbb{N}\}$, $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

But this contradicts the Nested Int. Prop.

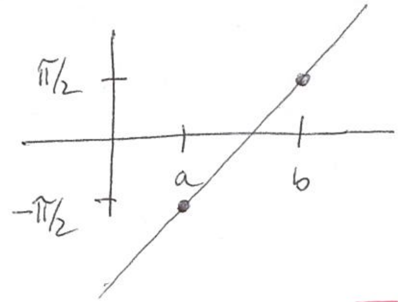
QED

Exercise 1.5.4

any open interval

(a) Show $\boxed{(a,b) \sim \mathbb{R}}$ (i.e. there is a bijection $f : (a,b) \xrightarrow{\sim} \mathbb{R}$).

Soln: Consider the line



$m = \dfrac{\pi/2 - (-\pi/2)}{b-a} = \dfrac{\pi}{b-a}$

$P = (a, -\pi/2)$

$\Longrightarrow \quad y + \pi/2 = \dfrac{\pi}{b-a}(x-a)$

$\Longrightarrow \quad \boxed{y \equiv f_1(x) = \dfrac{\pi}{b-a}(x-a) - \pi/2}$

Clearly, since f_1 is a line $f_1(x) = mx+b$, it is bijective $\left(y = mx+b \iff x = \frac{y-b}{m} = f^{-1}(y) \right)$.

Compose this function with the tangent function:

$$f_2(x) = \tan x$$

as

$$\boxed{\begin{aligned} f(x) &:= (f_2 \circ f_1)(x), \qquad f : (a,b) \xrightarrow{\sim} \mathbb{R} \\ &= \tan\left(\frac{\pi}{b-a}(x-a) - \frac{\pi}{2} \right) \end{aligned}}$$

and observe that since $\tan x$ is bijective (why?), f is also bijective.

QED

(b) $\boxed{(a, \infty) \sim \mathbb{R}}$

pf: Let $\boxed{f : (a,\infty) \to \mathbb{R} \text{ be } f(x) = \ln(x-a)}$

QED

(c) Show $\boxed{[0,1) \sim (0,1).}$

Soln:

$$\boxed{f(x) = \begin{cases} 0, & \text{if } x = 1/2 \\ 2x, & \text{if } x = \frac{1}{2^k}, \ k \geq 2 \\ x, & \text{else} \end{cases}}$$

Thm. 1.6.1 Cantor's Diagonalization pf that $(0,1) \not\sim \mathbb{N}$ (or $\mathbb{R} \not\sim \mathbb{N}$).

Thm. 1.6.2 (Cantor's Thm.) Given any set A, no function

$$f: A \to \mathcal{P}(A)$$

can be onto. power set of A

pf: Let $B = \{a \in A \mid a \notin f(a)\}$. If f were onto, then since $B \subseteq A$, $\exists\, a' \in A$ s.t.

$$f(a') = B$$

Either $a' \in B$ or $a' \notin B$.

(1) $a' \in B \implies a' \in \{a \in A \mid a \notin f(a)\}$

$\implies a' \notin f(a') = B$

so $a' \in B$ & $a' \notin B$ contrad.!

(2) $a' \notin B \implies a' \in f(a') = B$ contrad.!

QED