7/9/20

Prop.: If $S \subseteq \mathbb{R}$ is bdd. below, then

$$\inf S = -\sup(-S)$$

exists.

pf: $-S = \{-s \mid s \in S\}$, which, since S is bdd below,

$$S^{\ell} \neq \emptyset \text{ i.e. } \exists\, \ell \leq s \text{ for all } s \in S$$

we have

$$\exists\, -\ell \geq -s \text{ for all } s \in S \quad (\text{i.e. all } -s \in -S)$$

so

$$(-S)^{u} \neq \emptyset \quad (\text{bec. } -\ell \in (-S)^{u})$$

& therefore

$\sup(-S)$ exists by the completeness axiom

Now, to see that $-\sup(-S) = \inf S$, note that if $l \in S^l$ is any other lower bound for S, then $-l \in (-S)^u$, so

$$\sup(-S) \leq -l$$

$$\Longrightarrow \quad l \leq -\sup(-S)$$

$$\Longrightarrow \quad -\sup(-S) = \max(S^l) \stackrel{\text{def}}{=} \inf S$$

QED

Prop 2 $\sup S$ and $\inf S$ are unique, if they exist.

pf: By Prop. 1 it suffices to prove $\sup S$ is unique. If a & b are both least upper bounds, then they are in particular upper bounds. Using this fact, since $a = \min S^u$,

This really just says $\min S^u$ is unique.

$$\left.\begin{array}{l} a \leq b \\ \text{\& since } b = \min S^u \text{ too, } b \leq a. \end{array}\right\} \Longrightarrow a = b.$$

QED

<u>Lemma 1.3.8</u> For any $A \subseteq \mathbb{R}$ with $A^u \neq \emptyset$, a given upper bound $s \in A$ is the supremum of A

$$\boxed{\begin{gathered} s = \sup A \\ \text{if and only if} \\ \forall \varepsilon > 0, \ \exists a \in A, \quad s - \varepsilon < a \end{gathered}}$$

<u>pf:</u> (1) $s = \sup A \implies s = \min A^u$

$\implies s - \varepsilon \notin A^u$

$\implies \exists a \in A, \quad s - \varepsilon < a$

(2) $(\forall \varepsilon > 0, \ \exists a \in A, \quad s - \varepsilon < a)$

$\implies s - \varepsilon \notin A^u$

$\implies \underbrace{\forall b \in A^u, \quad s \leq b}$

i.e. $s = \min A^u = \sup A$

pf: $b < s \implies 0 < s - b \overset{def}{=} \varepsilon \implies b = s - \varepsilon \notin A^u$

Ex. 1.3.6 Let $A, B \subseteq \mathbb{R}$ be bdd. above.

(a) Let $s = \sup A$
$t = \text{sub } B$

$\Rightarrow \forall a \in A, b \in B, \quad a \leq s$
$b \leq t$

$\Rightarrow \quad a + b \leq s + b$
$\leq s + t$

$\Rightarrow \quad s + t \in (A+B)^u$

(b) Let $u \in (A+B)^u$, fix $a \in A$.

$\Rightarrow \forall b \in B, \quad a + b \leq u$

$\Rightarrow$ since $b \leq t$, $a + t \leq u$

(if $a + t > u$ then $t > u - a$
$\Rightarrow u - a \notin B^u \Rightarrow b > u - a$
for some $b \in B \Rightarrow a + b > u$,
contrad.)

$(\Rightarrow \quad t \leq u - a)$

We can similarly fix $b \in B$ to conclude

$$s+b \leq u, \quad \forall b \in B$$

(c) So,

$$s+b \leq u, \quad \forall b \in B$$
$$a+t \leq u, \quad \forall a \in A$$

$$\Longrightarrow \quad b \leq u-s, \quad \forall b \in B$$
$$a \leq u-t, \quad \forall a \in A$$

$$\Longrightarrow \quad u-s \in B^u$$
$$u-t \in A^u$$

$$\Longrightarrow \quad u-s \geq t = \min B^u$$
$$u-t \geq s = \min A^u$$

$$\Longrightarrow \quad u \geq s+t, \quad \forall u \in (A+B)^u$$

$$\Longrightarrow \quad s+t = \min (A+B)^u$$
$$= \sup (A+B)$$

QED

(Least Upper Bd. Property $\longrightarrow$ Nested Interval Property)

Thm. 1.4.1 (Nested Interval Property of $\mathbb{R}$)

Consider a sequence of nested ~~...~~ closed intervals

$$I_n = [a_n, b_n]$$

with

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

i.e.

$$a_1 \leq a_2 \leq a_3 \leq \cdots \quad \leq b_3 \leq b_2 \leq b_1$$

Then

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

pf: Any b_n is an upper bound for the a_n's, & vice-versa:

- let $A := \{a_n \mid n \in \mathbb{N}\}$, $B := \{b_n \mid n \in \mathbb{N}\}$

then $B \subseteq A^u$ = upper bounds of A

$A \subseteq B^l$ = lower bounds of B

$\Longrightarrow$ A is bounded above & B is bdd below.

Since

$$\sup A = \min A^u$$

$$\inf B = \max B^l$$

exist by assump., we conclude

all $a_n \le \sup A$

all $b_n \ge \inf B$

& since

~~inf B ≤ sup A ≤~~

$$a_n \le \sup A \le \inf B \le b_n$$

we see that

$$\sup A,\ \inf B \in [a_n, b_n] = I_n$$

for all n.

Therefore $$\bigcap_{n=1}^{\infty} I_n \supseteq \{\sup A, \inf B\} \ne \emptyset$$

may be equal, or may not.

QED

Thm. 1.4.2 (Archimedean Property of $\mathbb{R}$)

(1) For any $x \in \mathbb{R}$ there exists an $n \in \mathbb{N}$ s.t.

$$x < n$$ (ℕ is not bdd above)

$$\mathbb{R} \cap \mathbb{N}^u = \emptyset$$

(2) For any $y > 0$, there exists an $n \in \mathbb{N}$ s.t.

$$0 < \frac{1}{n} < y$$

pf: (1) The key feature of $\mathbb{N}$ is that it possesses the successor function: $s(n) := n+1$.
Suppose $\mathbb{R} \cap \mathbb{N}^u \neq \emptyset$, & let $\alpha = \sup \mathbb{N}$.
By Lemma 1.3.8, $1 > 0 \Longrightarrow$

$$\alpha - 1 \notin \mathbb{N}^u$$

i.e.

$$\exists n \in \mathbb{N} \text{ w/ } \alpha - 1 < n$$

But then

$$\alpha < \underbrace{n+1 = s(n) \in \mathbb{N}}_{\text{by def. of } s \,\&\, \mathbb{N}}$$ contradiction!!!

(2) Let $x = 1/y$. By (1), $\exists n \in \mathbb{N}$ satisfying

$$x < n$$
$$\Rightarrow \frac{1}{y} < n$$
$$\Rightarrow \frac{1}{n} < y$$

QED

Thm. (Integer Part of a Real Number):

For any $x \in \mathbb{R}$ there exists a unique integer $n \in \mathbb{Z}$, called the integer part of x, denoted

$$\boxed{n = \lfloor x \rfloor}$$

& satisfying

$$\boxed{x - 1 < n \le x}$$

e.g.

$$\lfloor \pi \rfloor = 3 \quad \& \quad \pi - 1 < 3 \le \pi$$

pf: Since $x \in \mathbb{R}$, the Archimedean property of $\mathbb{R}$ applies to give us some $N \in \mathbb{N}$ with

$$x < N$$

& applying it to $-x$, some $M \in \mathbb{N}$ w/

$$-x < M \implies -M < x$$

i.e. $\boxed{-M < x < N}$

Define $S \subseteq \mathbb{N}$ by

$$S := \{ n \in \mathbb{N} \mid -M + \boxed{n} - 1 \leq x \}$$

& observe that

$$-M < x \implies -M + \boxed{1} - 1 = -M < x$$

$$\implies 1 \in S \text{ so } S \neq \emptyset$$

Therefore $\mathbb{N} - S$ has a minimum by the well-ordering property of $\mathbb{N}$, call it $n+1$:

$$n+1 = \min(\mathbb{N} - S)$$

$$\iff n+1 \notin S \text{ but } \underbrace{n \in S}$$

n is smaller than $n+1 = \min \mathbb{N} - S$

$$\iff -M + (n+1) - 1 = -M + n > x \quad \& \quad -M + n - 1 \leq x$$

$$\iff x \boxed{-1} < \underbrace{-M + n \boxed{-1}}_{= m \in \mathbb{Z}} \leq x$$

Call $m = -M + n + 1 \in \mathbb{Z}$ the integer part.

This shows existence of $m \in \mathbb{Z}$ with $x-1 < m \leq x$.

For uniqueness, supp. $m_1, m_2 \in \mathbb{Z}$ both satisfied

$$x-1 < m_i \leq x$$

~~Since~~ But $|m_1 - m_2| \in \mathbb{N}$ and

$$x-1 < m_1 \leq x$$
$$x-1 < m_2 \leq x$$

$$\Longrightarrow \quad |m_1 - m_2| < |x-(x-1)| = 1$$

$$\Longrightarrow \quad |m_1 - m_2| = 0$$

$$\Longrightarrow \quad m_1 = m_2$$

QED

Thm. 1.4.3 (Density of $\mathbb{Q}$ in $\mathbb{R}$)

For every distinct pair $a, b \in \mathbb{R}$ of real #'s, $a < b$

There exits an integer $r \in \mathbb{Q}$ s.t.,

$$a < r < b$$

pf: By Archimedes. (2), $\exists\, n \in \mathbb{N}$ s.t.

$$\boxed{\frac{1}{n} < b - a} \qquad (i)$$

Secondly, since $na \in \mathbb{R}$, Archim. (1) gives $m \in \mathbb{N}$ s.t.

$$na < m$$

and if m is the smallest such integer (which exists by the well ordering of $\mathbb{N}$) then

$$\boxed{m - 1 \le na < m} \qquad (ii)$$

The 1st ineq. in (ii) gives

$$m - 1 \le na \implies m \le na + 1 \qquad \text{(iii)}$$

while (i) gives

$$\frac{1}{n} < b - a \implies a < b - 1/n \qquad \text{(iv)}$$

Combining these we get

$$\begin{aligned} m &\le na + 1 \qquad \text{(iii)} \qquad \text{(v)} \\ &< n(b - 1/n) + 1 \\ &= nb - 1 + 1 \\ &= nb \end{aligned}$$

So

$$na \overset{\text{(ii)}}{<} m \le nb$$

$$\implies a < \underset{\underset{\mathbb{Q}}{\cap}}{\frac{m}{n}} \le b$$

QED

(Density of Irrationals in $\mathbb{R}$)

<u>Corollary 1.4.4</u> Given any $a, b \in \mathbb{R}$ w/

$$a < b$$

there exists an <u>irrational</u> # $t \in \mathbb{R} \setminus \mathbb{Q}$ s.t.

$$a < t < b$$

<u>pf:</u> $\frac{1}{\sqrt{2}} \notin \mathbb{Q}$ & by the density of $\mathbb{Q}$ in $\mathbb{R}$

$$\sqrt{2}a < \sqrt{2}b \implies \sqrt{2}a < r < \sqrt{2}b \quad \text{for some } r \in \mathbb{Q}$$

$$\implies a < \underbrace{\frac{r}{\sqrt{2}}}_{\notin \mathbb{Q}} < b$$

<u>QED</u>

bec. $r \in \mathbb{Q}$, $\frac{1}{\sqrt{2}} \notin \mathbb{Q}$

$\implies \frac{r}{\sqrt{2}} \notin \mathbb{Q}$