Connectedness

Def $A, B \subseteq \mathbb{R}$ are said to be separated if

$$\bar{A} \cap B = A \cap \bar{B} = \emptyset$$

If $E \subseteq \mathbb{R}$ can be written

$$\boxed{E = A \cup B}$$

w/ $A, B \neq \emptyset$ & separated, then E is called disconnected. Else, connected.

ex. 1 $A = (1,2)$, $B = (3,5)$ separated

ex. 2 $A = (1,2]$, $B = (2,5)$ not separated

ex. 3 $\mathbb{Q}$ is disconnected, since

$$\mathbb{Q} = \underbrace{\left(\mathbb{Q} \cap (-\infty, \sqrt{2})\right)}_{A} \cup \underbrace{\left(\mathbb{Q} \cap (\sqrt{2}, \infty)\right)}_{B}$$

$$\bar{A} \cap B = (-\infty, \sqrt{2}] \cap B = \emptyset$$

Thm. 3.4.6 $E \subseteq \mathbb{R}$ is connected $\iff$

$\forall A, B \neq \emptyset$ w/ $E = A \cup B$, $A \cap B = \emptyset$,

$\exists x_n \to x$ w/

$x_n \in A$, $x \in B$ i.e. $B \cap L(A) \neq \emptyset$

OR

$x_n \in B$, $x \in A$ i.e. $A \cap L(B) \neq \emptyset$

pf: $\Rightarrow$: E connected $\Rightarrow \nexists A, B \neq \emptyset$

& separated s.t. $A \cup B = E \Rightarrow \nexists A, B \neq \emptyset$

s.t.

$\bar{A} \cap B = \emptyset$, $A \cap \bar{B} = \emptyset$

$\Rightarrow$ since $A \subseteq \bar{A}$, $B \subseteq \bar{B}$,

$\emptyset \subseteq A \cap B \subseteq \bar{A} \cap B = \emptyset$, say

$\Rightarrow A \cap B = \emptyset$

I.e., if $A, B \neq \emptyset$ & $E = A \cup B$,

then $\bar{A} \cap B \neq \emptyset$ or $A \cap \bar{B} \neq \emptyset$. Say

$\bar{A} \cap B \neq \emptyset$, so $(A \cup L(A)) \cap B = \overset{\emptyset}{\cancel{(A \cap B)}} \cup (L(A) \cap B)$

$= L(A) \cap B \neq \emptyset$

$\Rightarrow \exists x \in L(A) \cap B$

$\Rightarrow \exists x_n \in A - \{x\}, \ x_n \to x \in B.$

Conversely, if $\forall A, B \neq \emptyset$ w/ $E = A \cup B$, $A \cap B = \emptyset$

$\exists x_n \to x$ w/

$$x_n \in A, \ x \in B$$

or

$$x_n \in B, \ x \in A$$

then, since $A \cap B = \emptyset$, say $x_n \in A$, $x \in B$,

$x_n \neq x$, so $x \in L(A)$

$$\Rightarrow x \in L(A) \cap B = \underbrace{(A \cap B)}_{= \emptyset} \cup (L(A) \cap B)$$

$$= (A \cup L(A)) \cap B$$

$$= \bar{A} \cap B$$

$$\neq \emptyset \quad !$$

$\Rightarrow A, B$ are ~~separated~~ not separated

$\Rightarrow E$ is connected. QED

<u>Prop.</u> Disjoint open sets A, B are separated,

$$A, B \in \mathcal{T} \ \& \ A \cap B = \emptyset$$

$$\Longrightarrow \overline{A} \cap B = A \cap \overline{B} = \emptyset$$

<u>pf:</u> Since A, B are open, $A = A^\circ$, $B = B^\circ$; so along with $A \cap B = \emptyset$ we have

$$A \subseteq B^c, \quad B \subseteq A^c$$

$$\Longrightarrow \overline{A} \subseteq \overline{B^c} = (B^\circ)^c = B^c \qquad \overline{B} \subseteq \overline{A^c} = (A^\circ)^c = A^c$$

$$\Longrightarrow \overline{A} \cap B = \emptyset \qquad \Longrightarrow \overline{B} \cap A = \emptyset.$$

<u>QED</u>

<u>Thm.</u> $E \subseteq \mathbb{R}$ is connected $\Longrightarrow$ E cannot be written as the disjoint union of two open sets,

$$E \neq A \cup B, \quad \forall A, B \in \mathcal{T}, \quad A \cap B = \emptyset$$

<u>pf:</u> If E is connected, then

$$E \neq A \cup B, \ \forall A, B \text{ separated, i.e. } \overline{A} \cap B = A \cap \overline{B} = \emptyset,$$

But disjoint open sets $A, B \in \mathcal{T}$, $A \cap B = \emptyset$, are separated, so $E \neq A \cup B$ for such sets.

~~Conversely, if E cannot be written as the union of disjoint open sets, and A, B are separated, then~~ $\cancel{E \neq A^\circ \cup B^\circ}$, ~~since~~

$$\cancel{\left.\begin{array}{l} A^\circ \subseteq A \Rightarrow \overline{A^\circ} \subseteq \overline{A} \\ B^\circ \subseteq B \Rightarrow \overline{B^\circ} \subseteq \overline{B} \end{array}\right\} \Rightarrow}$$

<u>Thm.</u> If $E \subseteq \mathbb{R}$ is connected, then $\overline{E}$ is also connected.

<u>pf:</u> $\overline{E}$ disconnected $\Longrightarrow$

$$\overline{E} = A \cup B, \quad A, B \neq \emptyset$$
$$\overline{A} \cap B = A \cap \overline{B} = \emptyset$$

$$\Longrightarrow \quad E = E \cap (A \cup B)$$
$$= (E \cap A) \cup (E \cap B)$$

and

$$(\overline{E \cap A}) \cap (E \cap B)$$
$$\subseteq \overline{A} \cap B = \emptyset$$

&

$$(E \cap A) \cap (\overline{E \cap B})$$
$$\subseteq A \cap \overline{B} = \emptyset$$

$$\Longrightarrow (\overline{E \cap A}) \cap (E \cap B) = \emptyset$$
$$(E \cap A) \cap (\overline{E \cap B}) = \emptyset$$

Finally we observe that neither $E \cap A$ nor $E \cap B = \emptyset$, for if, say, $E \cap A = \emptyset$, then

$$\emptyset \subsetneq E = E \cap B \subseteq B$$

$$\Longrightarrow \quad A \cup B = \overline{E} \subseteq \overline{B}$$

$$\Longrightarrow \quad A = A \cap (A \cup B)$$

$$\subseteq A \cap \overline{B} = \emptyset$$

$$\Longrightarrow \quad A = \emptyset$$ contradiction.

QED

Thm. 3.4.7 A set $E \subseteq \mathbb{R}$ is connected

$$\iff \forall a, b \in E, \quad \left(a < b \Rightarrow (a,b) \subseteq E \right)$$

pf: Let E be connected, $a, b \in E$ & $a < b$. To see that $(a,b) \subseteq E$, let $c \in (a,b)$, i.e.

$$a < c < b$$

and let us show $c \in E$. Define

$$A = (-\infty, c) \cap E, \quad B = (c, \infty) \cap E$$

Then, $\bar{A} \cap B = (-\infty, c] \cap E \cap (c, \infty) \cap E = \emptyset$ &

$$A \cap B = (-\infty, c) \cap E \cap [c, \infty) \cap E = \emptyset$$

so if $E = A \cup B$, then E would be disconnected, contradiction. I.e. $E - \{c\}$ is disconnected + $E - \{c\} \subseteq E \Rightarrow c \in E$.

Conversely, if $\forall a, b \in E$, $(a < b \Rightarrow (a,b) \subseteq E)$, and if

$$E = A \cup B, \quad A, B \neq \emptyset, \quad A \cap B = \emptyset$$

then, to show E is connected we must show

$$\bar{A} \cap B \neq \emptyset$$

or

$$A \cap \bar{B} \neq \emptyset$$

or equivalently, by Thm. 3.4.6,

$$B \cap L(A) \neq \emptyset$$

or

$$A \cap L(B) \neq \emptyset$$

Let $a_0 < b_0$ for $a_0, b_0 \in E$. Then

$a_0 \in A,\ b_0 \in B$

$$a_0, b_0, (a_0, b_0) \subseteq E \Rightarrow \overbrace{[a_0, b_0]}^{= I_0} \subseteq E$$

Let $c_0 = \frac{a_0 + b_0}{2}$ & note either $c_0 \in A$ or $c_0 \in B$.

If $c_0 \in A$, let

$$I_1 = [c_0, b_0] =: [a_1, b_1]$$

(so ~~c_0~~ $c_0 = a_1 \in A$, $b_1 \in B$)

& if $c_0 \in B$, let

$$I_1 = [a_0, c_0] =: [a_1, b_1]$$

(so $a_0 = a_1 \in A$, $c_0 = b_1 \in B$)

Repeat.

$$I_0 \supseteq I_1 \supseteq \cdots$$

$$\lim_{n\to\infty} b_n - a_n = 0$$

$$\Longrightarrow \bigcap_{n=1}^{\infty} I_n = \{x\} \neq \emptyset$$

and

$$a_n \to x$$

$$b_n \to x$$

Since $x \in E = A \cup B \Rightarrow x \in A$ or $x \in B$,

& if ~~say~~ $x \in A$, then $x \in L(B) \cap A \neq \emptyset$,

& if $x \in B$, then $x \in L(A) \cap B \neq \emptyset$. QED

Corollary: A set $E \subseteq \mathbb{R}$ is connected $\Longleftrightarrow$ E is an interval,

(i) $E = (a,b)$ or (a,∞) or $(-\infty,b)$ $\in \mathcal{I}_o$

(ii) $E = [a,b] \in \mathcal{I}_c$

(iii) $E = (a,b]$ or $(-\infty,b]$ or $[a,b)$ or $[a,\infty) \in \mathcal{I}_{oc} \cup \mathcal{I}_{co}$

pf: E is connected iff $\forall a,b \in E$

$$a<b \Rightarrow (a,b) \subseteq E$$

iff

$$\bigcup_{\substack{a<b \\ a,b\in E}} (a,b) \subseteq E$$

If E is bounded above, then $\underset{b\,=}{\sup E} = \min E^u$ exists, so either $b = \sup E \in E$ or $\notin E$, dep. on whether $E = (a,b)$ or $(a,b]$ or $(-\infty,b)$ or $(-\infty,b]$ etc.