Compactness

Prop. 1 If $K \subseteq \mathbb{R}$ compact & $A \subseteq K$ is closed, then A is also compact.

pf: Let $\mathcal{U} = \{O_\lambda \mid \lambda \in \Lambda\}$ be an open cover of A. Since A is closed, A^c is open & $K - A \subseteq \mathbb{R} - A = A^c$. Then,

$$\mathcal{U} \cup \{A^c\}$$

covers K,

$$K \subseteq \underbrace{\left(\bigcup_{\lambda \in \Lambda} O_\lambda\right)}_{\text{cover } A} \cup \underbrace{A^c}_{\text{covers } K-A}$$

$\Rightarrow$ $\mathcal{U} \cup \{A^c\}$ may be reduced to a finite subcover of K,

$$\Rightarrow \quad A \subseteq K \subseteq \bigcup_{k=1}^{n} O_{\lambda_k} \in \mathcal{U} \cup \{A^c\}.$$

QED

<u>Heine–Borel Thm.</u> Let $K \subseteq \mathbb{R}$. Then

TFAE:

(1) K is <u>closed & bdd.</u>

(2) K is <u>sequentially cpt.</u> (every seq. in K has a convergent subseq. w/ limit in K)

(3) K is <u>compact</u> (every open cover has a finite subcover)

<u>pf:</u> (1) $\Rightarrow$ (2) (Ex. 3.3.3) The proof is the same as for a closed interval: since K is bdd., ~~$\exists M > 0$ s.t.~~

$\cancel{|x| \leq M, \ \forall x \in K}$

$\cancel{\iff K \subseteq [-M, M] = I_1}$

~~Let I_2 be the half $[-M, 0] \cap K$ or $[0, M] \cap K$~~

any sequence $(x_n)_{n \in \mathbb{N}}$ in K is also bdd.,

so has a convergent subseq.

$$x_{n_k} \longrightarrow x$$

by B.W. Since $x_{n_k} \in K$, $\forall k$ & K is closed,

$x \in K$. Thus, K is sequentially cpt.

<u>(2) $\Longrightarrow$ (1):</u> If K is sequentially cpt.,

every seq. x_n in K has a convergent subseq.

w/ limit in K. In particular, let

$$x \in L(K)$$

& $\quad x_n \in K - \{x\}$ a seq. $x_n \to x$. (Thm. 3-2.5)

Then every subseq. also converges to x, & $x \in K$

for all of them, so $x \in K$. Thus $L(K) \subseteq K$

& K is closed. Suppose K were <u>unbounded</u>

Then $(k_n)_{n\in\mathbb{N}}$ is a seq. in K with no convergent subseq. (bec. convergent sequences are bounded), so K is not sequentially compact. This is the contrapositive of $(1)\Longrightarrow(3)$ as far as boundedness

[illegible]

To show closedness, let x_n be a convergent seq. in K with limit $x = \lim_{n\to\infty} x_n$. Since K is sequentially cpt., x_n has a convergent subseq. $x_{n_k}\longrightarrow x'$ with $x'\in K$. But $x'=x$ by Thm. 2.5.2.

cf. p. ② alternative $(2)\Rightarrow(1)$

<u>$(3)\Longrightarrow(1)$:</u> Suppose K is closed & bdd. & let a_n be a seq. in K. By Bolzano-Weierstrass, a_n has a convergent subseq. $a_{n_k}\to a$ in K. Since K is closed, $a\in K$.

i.e. $= \neg(\exists M>0, \forall k\in K, |k|\le M)$

$(\forall M>0, \exists x\in K$ w/ $|x|>M)$

Create a seq.

$M=1 \Rightarrow \exists x_1\in K, \quad |x_1|>M=1$

$M=2 \Rightarrow \exists x_2\in K, \quad |x_2|>M=2$

$\vdots$

$M=n \Rightarrow \exists x_n\in K, \quad |x_n|>n$

The seq., being in K, has a convergent subseq. x_{n_k}, but this contradicts x_{n_k}'s unboundedness. Thus, K must be bdd.

Ex. (3.3.9) <u>(2) ⇒ (3)</u> Let K be sequentially cpt. & use the same argument as in the closed interval case: let $\mathcal{U} = \{O_\lambda \mid \lambda\in\Lambda\}$ be an arbitrary open cover of K. From

(1) $\Leftrightarrow$ (2) we know K is bounded, say by M:

$$K \subseteq [-M, M] \overset{def}{=}: I_1$$

Suppose $\mathcal{U}$ had no finite subcover. Then,

One of the two halves

$$K \cap [-M, 0]$$

or

$$K \cap [0, M]$$

cannot possess a finite subcover, call

$$I_2 = [-M, 0] \text{ or } [0, M]$$

(whichever $\cap K$ has no finite subcover)

Repeat. Then

$$I_1 \supseteq I_2 \supseteq \cdots$$

$$\Rightarrow \bigcap_{n=1}^{\infty} I_n = \{x\} \neq \emptyset$$

(bc. $\lim_{n\to\infty} (b_n - a_n) = \lim_{n\to\infty} \frac{2M}{2^{n-1}} = 0$)

But then $x \in K \subseteq \bigcup_{\lambda\in\Lambda} O_\lambda \Rightarrow x \in O_\lambda$ for some λ

$\Rightarrow \exists \varepsilon > 0, V_\varepsilon(x) \subseteq O_\lambda$

$\Rightarrow$ for this $\varepsilon > 0$,

since $(b_n - a_n) \to 0$, $\exists N \in \mathbb{N}$,

$n \geq N \Rightarrow b_n - a_n < \varepsilon/2$

$\Rightarrow I_n = [a_n, b_n] \subseteq V_\varepsilon(x) \subseteq O_x$

($x \in [a_n, b_n]$!)

$\Rightarrow I_n \cap K$ has a finite subcover, $\{O_x\}$, of $\mathcal{U}$ contradiction.

<u>(3) $\Rightarrow$ (2):</u> Let K be cpt, but supp. K were <u>not</u> seq. cpt. Then, there is a seq. x_n in K w/ no convergent subseq. in K. The set

$$S = \{x_n \in K \mid n \in \mathbb{N}\} \subseteq K$$

has no limit pts. (it has only isolated pts), & so no convergent non-constant seq.. I.e.

$$L(S) = \emptyset \subseteq S \text{ trivially}$$

So <u>S is closed</u>, but <u>not</u> bounded! Why? (B.W.)

Can also construct an open cover $\mathcal{U}$ of S s.t. $x_n \in U_n$ but $U_i \cap U_j = \emptyset$, if $i \neq j$, so $\mathcal{U}$ has no finite subcover, contradiction.

(Lemma)

<u>Ex. 3.3.8</u> Let $K, L \subseteq \mathbb{R}$ be nonempty & compact, & define

$$\boxed{d \stackrel{\text{def}}{=} \inf\{|x-y| \mid x \in K,\ y \in L\}} = d(K,L)$$

be the <u>distance</u> bet. K & L. Then,

$$\boxed{K \cap L = \emptyset \implies d = |x_0 - y_0| > 0 \quad \text{for some } x_0 \in K,\ y_0 \in L}$$

<u>pf:</u> If $d = 0$, then, $\forall \varepsilon_n > 0$ $\exists x_n \in K$, $y_n \in L$ s.t.

$$|x_n - y_n| < \varepsilon_n = \frac{1}{n} \text{ say}$$

and $\varepsilon_n \to 0$.

Since K is compact, it is sequentially compact, so $\exists a_{n_k}$ subseq. of a_n with limit $a = \lim_{k\to\infty} a_{n_k}$. Then,

$$|b_{n_k} - a| \le \underbrace{|b_{n_k} - a_{n_k}|}_{< \varepsilon_{n_k}/2} + \underbrace{|a_{n_k} - a|}_{< \varepsilon_{n_k}/2 \text{ too}} < \varepsilon_{n_k}$$

But this means

$$a = \lim_{k\to\infty} b_{n_k} \in L$$

& $a \in K$ already, so

$$a \in K \cap L \implies K \cap L \neq \emptyset$$

<u>contradiction</u>.

Hence, $d = d(K, L) > 0$. <u>QED</u>

Bonus: proof of $(1) \Longleftrightarrow (3)$

$(3) \Longrightarrow (1)$

~~(1) ⇒ (3)~~: Suppose K is compact, & let us first show K is bounded. Consider the open cover

$$\mathcal{U} \stackrel{\text{def}}{=} \{ \underbrace{V_1(x)}_{\varepsilon = 1\text{-neighborhood}} \mid x \in K \}$$

$$V_1(x) = (x-1, x+1) \in \mathcal{T}$$

By compactness of K we can reduce this to a finite subcover

$$\mathcal{U}_n \stackrel{\text{def}}{=} \{ V_1(x_1), \ldots, V_1(x_n) \}$$

$$(x_1, \ldots, x_n \in K)$$

i.e.

$$K \subseteq V_1(x_1) \cup V_1(x_2) \cup \cdots \cup V_1(x_n)$$

But if

$$M = \max\{|x_1|, \ldots, |x_n|\}$$

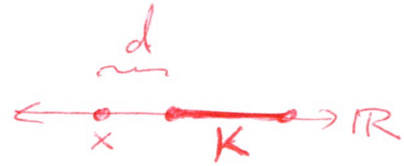
then

$$|k| \le M + 1, \quad \forall k \in K$$

so K is bounded.

K is also closed: let $x \in \mathbb{R} - K$ & observe that

- $\{x\}$ is compact
- K is compact



$\Longrightarrow$ (Ex. 3.3.8) $\quad d(\{x\}, K) = d > 0$

$\Longrightarrow$ $\quad V_{d/2}(x) \subseteq (\mathbb{R} - K)$ covers x

$\Longrightarrow$ $\quad \underbrace{\mathbb{R} - K}_{= K^c}$ is open

$\Longrightarrow$ K is closed. $\qquad$ QED

Thm. 3.3.5 (**Nested Compact Sets Property**)

Let $K_n \in \mathcal{K}_{\mathbb{R}}$ be nested

$$K_1 \supseteq K_2 \supseteq \cdots$$

Then,
$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset$$

pf: Let $x_n \in K_n \subseteq K_1$ for ea. $n \in \mathbb{N}$. Since K_1 is cpt., it is sequentially cpt. $\Longrightarrow$ $\exists\, x_{n_k}$ subsequence converging, say, to $x \in K_1$.

But this argument applies to each K_n, for

$$\begin{cases} x_{n_k} \in K_{n_k} \\ \& \\ x_{n_k} \longrightarrow x \end{cases}$$

$$\Longrightarrow \underline{\forall N \in \mathbb{N},} \quad n_k \geq N \Longrightarrow x_{n_k} \in K_N$$

$$\Longrightarrow \underline{x \in K_N} \Longrightarrow x \in \bigcap_{n=1}^{\infty} K_n \neq \emptyset$$

QED