(24) $\boxed{L(A) \in \mathcal{C}, \quad \forall A \subseteq \mathbb{R}.}$

Pf: We must show $L(L(A)) \subseteq L(A)$:

Let $x \in L(L(A))$, then

$$\forall \varepsilon > 0, \quad V_\varepsilon(x) \cap \left(L(A) - \{x\}\right) \neq \emptyset$$

$$\Longrightarrow \forall \varepsilon > 0, \ \exists y \in V_\varepsilon(x) \cap L(A), \ y \neq x$$

$$\Longrightarrow \forall \varepsilon > 0, \ \exists y \in L(A), \ |x-y| < \varepsilon, \ y \neq x$$

$$\Longrightarrow \forall \varepsilon > 0, \ \forall \delta > 0, \ \exists y \in \mathbb{R},$$

- $|x-y| < \varepsilon$
- $V_\delta(y) \cap (A - \{y\}) \neq \emptyset$
- $x \neq y$

$$\Longrightarrow \forall \varepsilon > 0, \ \forall \delta > 0, \ \exists y \in \mathbb{R}, \ \exists z \in A$$

- $x \neq y, \ y \neq z$
- $|x-y| < \varepsilon$
- $|y-z| < \delta$

Taking $\delta = \varepsilon/2$ & replacing ε with $\varepsilon/2$,

& using triangle ineq., we have $x \in L(L(A)) \Longrightarrow$

$$\forall \varepsilon > 0,\ \exists z \in A,\ x \neq z$$

$$\underbrace{\begin{aligned} |x-z| &= |x-y+y-z| \quad (\exists y \in L(A),\ \text{recall}) \\ &\leq |x-y| + |y-z| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \end{aligned}}_{\text{says } z \in V_\varepsilon(x) \cap (A - \{z\}) \ \Rightarrow\ V_\varepsilon(x) \cap (A - \{z\}) \neq \emptyset}$$

$\Longrightarrow x \in L(A)$

Thus, $L(L(A)) \subseteq L(A)$ & $L(A)$ is closed.

~~Remarks: If $x \in L(A)$, the above proof is unnecessary, so assume that $x \notin L(A)$, but $x \in L(L(A))$ (this may be unnecessary, in actual fact, but only so a posteriori, after we have proven that $L(L(A)) = L(A)$. Right now all we know is $L(A) \subseteq L(L(A))$.)~~

(has no isolated pts., why?)

(6) $\boxed{\overline{A} \overset{\text{def}}{=} A \cup L(A) \overset{\text{thm}}{=} \bigcap_{A \subseteq C \in \mathcal{C}} C \overset{\text{def}}{=:} X}$

pf: <u>Lemma</u> $L(A \cup L(A)) = L(A) \in \mathcal{C}$

i.e. $L(\overline{A}) = L(A) \in \mathcal{C}$

pf: $x \in L(A \cup L(A)) \iff$

$\forall \varepsilon > 0,\ \exists y \in V_{\varepsilon/2}(x) \cap ((A \cup L(A)) - \{x\})$

$\iff \forall \varepsilon > 0,\ \exists y \in A \cup L(A),$

$x \neq y,\ |x - y| < \varepsilon/2$

<u>case 1:</u> $y \in A \implies x \in L(A)$

<u>case 2:</u> $y \in L(A) \implies \forall \varepsilon > 0,$

$\exists z \in V_{\varepsilon/2}(y) \cap (A - \{y\})$

$\implies \forall \varepsilon > 0,\ \exists z \in (A - \{x\}),$

$$
\begin{aligned}
|x - z| &= |x - y + y - z| \\
&\leq |x - y| + |y - z| \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon
\end{aligned}
$$

$\implies x \in L(A).$

This shows $L(A \cup L(A)) \subseteq L(A)$. To show $L(A \cup L(A)) \supseteq L(A)$, observe that $A \subseteq A \cup L(A)$

$$\Rightarrow L(A) \subseteq L(A \cup L(A)).$$

QED

pf of (6):

(1) $\bar{A} \subseteq X$: $x \in \bar{A} = A \cup L(A)$

$$\Rightarrow x \in A \text{ or } x \in L(A)$$

$$\Rightarrow \forall C \in \mathcal{C},\ A \subseteq C,$$
$$x \in A \subseteq C \text{ or } x \in L(A) \subseteq L(C) \subseteq C$$

$$\Rightarrow \forall C \in \mathcal{C},\ A \subseteq C,\ x \in C$$

$$\Rightarrow x \in \bigcap_{A \subseteq C \in \mathcal{C}} C = X$$

(2) $\bar{A} \supseteq X$: By the lemma, $L(\bar{A}) = L(A) \subseteq A \cup L(A) = \bar{A}$

$\Longrightarrow \bar{A} \in \mathcal{C}$, and $A \subseteq A \cup L(A) = \bar{A}$

so $\bar{A} \supseteq X = \bigcap_{A \subseteq C \in \mathcal{C}} C$

<u>QED</u>

(6.5) $$\boxed{A^{\circ} = \bigcup_{\substack{O \subseteq A \\ O \in \mathcal{T}}} O} \stackrel{\text{def}}{=} X$$

pf: $A^{\circ} \stackrel{\text{def}}{=} \{a \in A \mid a \text{ is interior to } A\}$

$= \{a \in A \mid \exists \varepsilon_a > 0,\ V_{\varepsilon_a}(a) \subseteq A\}$

$$\Longrightarrow \quad A^{\circ} = \bigcup_{\substack{V_{\varepsilon_a}(a) \subseteq A \\ a \in A^{\circ} \\ V_{\varepsilon}(a) \in \mathcal{T}}} V_{\varepsilon_a}(a) \subseteq \bigcup_{\substack{O \subseteq A, \\ O \in \mathcal{T}}} O = X$$

Conversely, $\underset{\text{open}}{O} \subseteq A \Rightarrow O \subseteq A^{\circ}$ since

$$\left\{ \begin{array}{l} x \in O \subseteq A \Rightarrow x \in A \\ \quad \& \\ \exists \varepsilon > 0,\ V_{\varepsilon}(a) \subseteq O \subseteq A \end{array} \right\}$$

Therefore,

$$X = \bigcup_{\substack{O \subseteq A \\ \text{open}}} O \subseteq A^{\circ}$$

QED

(7) $A \in \mathcal{C} \iff A = \overline{A}$

$A \in \mathcal{T} \iff A = A^\circ$

pf: (1) If $A \in \mathcal{C}$, then since $A \subseteq A$ & $\overline{A} \overset{(6)}{=} \bigcap_{A \subseteq C \in \mathcal{C}} C$, we have

$$\left.\begin{array}{l} \overline{A} \subseteq A \\ \text{But also} \\ A \subseteq A \cup L(A) = \overline{A}, \end{array}\right\} \Rightarrow A = \overline{A}$$

so $\boxed{A \in \mathcal{C} \Rightarrow A = \overline{A}.}$

Conversely, if $A = \overline{A}$ then by 6) & (4) $A \in \mathcal{C}$, so $\boxed{A = \overline{A} \Rightarrow A \in \mathcal{C}}$

(2) If $A \in \mathcal{T}$, then all $a \in A$ are interior points of A, i.e. $a \in A^\circ$. Thus,

$$A \subseteq A^\circ$$

But $A^\circ \subseteq A$ by def. of A°, so

$$\boxed{A \in \mathcal{T} \implies A = A^\circ}$$

Conversely, $A^\circ = A \implies A \in \mathcal{T}$ bec. by (6.5) & (1) $A^\circ = \bigcup_{\substack{O \subseteq A \\ \text{open}}} O \in \mathcal{T}$.

QED