Topology of $\mathbb{R}$

Def. 1 ε-neighborhood of $a \in \mathbb{R}$:

$$\boxed{V_\varepsilon(a) = (a-\varepsilon,\ a+\varepsilon)}$$

Def. 2 open subset $O \subseteq \mathbb{R}$: "O is open" $\underset{=}{\text{def}}$

$$\boxed{\forall a \in O,\ \exists \varepsilon > 0 \text{ s.t. } V_\varepsilon(a) \subseteq O}$$

Such an $a \in A$ is called an interior point of A

$\mathcal{T} \overset{\text{def}}{=}$ set of all open subsets of A ($\mathcal{T}$ for "topology")

Def. 3 limit point of a subset $A \subseteq \mathbb{R}$:

"x is a limit point of A" $\overset{\text{def}}{=}$ $\forall \varepsilon > 0,\ V_\varepsilon(a) \cap (A - \{x\}) \neq \emptyset$

L or $L(A)$ $\overset{\text{def}}{=}$ set of all limit pts of A

Def. 4 isolated point of a subset $A \subseteq \mathbb{R}$

"$a \in A$ is isolated" $\overset{\text{def}}{=}$ $a \in A$ but $a \notin L(A)$

I or $I(A)$ $\overset{\text{def}}{=}$ set of all isolated pts of A

Def. 5 closed subset $A \subseteq \mathbb{R}$:

"$A \subseteq \mathbb{R}$ is closed" $\overset{\text{def}}{=}$ $L(A) \subseteq A$

<u>Def 6</u> <u>closure</u> of a subset $A \subseteq \mathbb{R}$:

$$\boxed{\bar{A} \overset{\text{def}}{:=} A \cup L(A)}$$

<u>Def 7</u> <u>closure</u> of a subset $A \subseteq \mathbb{R}$:

$\bar{A} \overset{\text{def}}{:=}$ "the smallest (with respect to set inclusion $\subseteq$ on $\mathcal{P}(A)$) closed subset containing A"

$$= \bigcap_{\substack{A \subseteq C \\ C \text{ closed}}} C$$

(the <u>intersection</u> of all closed $C \subseteq \mathbb{R}$ s.t. $A \subseteq C$)

<u>Thm. 3.1.12 (Abbott)</u> Def 6 is equivalent to Def 7. (pf below)

Def. 8 The interior of a subset $A \subseteq \mathbb{R}$:

$$\boxed{A^\circ \overset{\text{def}}{=} \text{the set of all interior points of } A \text{ (those } a \in A \text{ for which } \exists \varepsilon > 0 \text{ s.t. } V_\varepsilon(a) \subseteq A)}$$

Def. 9 boundary point of A:

"$a \in \mathbb{R}$ is a boundary point" $\overset{\text{def}}{=}$ $\forall \varepsilon > 0$, $V_\varepsilon(a) \cap A \neq \emptyset$ & $V_\varepsilon(a) \cap A^c \neq \emptyset$

$(A^c \overset{\text{def}}{=} \mathbb{R} - A)$

$\partial A \overset{\text{def}}{=}$ set of all bdry points of A $\overset{\text{def}}{=}$ boundary of A

Def. 10 adherent point of $A \subseteq \mathbb{R}$:

"a is an adherent point of A" $\overset{\text{def}}{=}$ $\forall \varepsilon > 0$, $V_\varepsilon(a) \cap A \neq \emptyset$

$A(A) \overset{\text{def}}{=}$ set of all adherent pts. of A

Theorems

open sets & $\cup, \cap$

(1) $\{O_i \in \mathcal{T} \mid i \in I\} \subseteq \mathcal{T} \Rightarrow \bigcup_{i \in I} O_i \in \mathcal{T}$ (Thm. 3.2.3)

(2) $\{O_1, \ldots, O_n\} \subseteq \mathcal{T} \Rightarrow O_1 \cap O_2 \cap \cdots \cap O_n \in \mathcal{T}$

(but not nec. $\bigcap_{i \in I} O_i \in \mathcal{T}$)

ex. $O_i = (0, 1 + 1/i)$

open vs. closed

(3) $O \in \mathcal{T} \iff O^c \in \mathcal{C} \stackrel{\text{def}}{=}$ all closed sets (Thm. 3.2.13)

$C \in \mathcal{C} \iff C^c \in \mathcal{T}$

(Thm. 3.2.14)

closed sets & $\cup, \cap$

(4) $C_1, \ldots, C_n \in \mathcal{C} \Rightarrow C_1 \cup C_2 \cup \cdots \cup C_n \in \mathcal{C}$

(but not nec. $\bigcup_{i \in I} C_i \in \mathcal{C}$, ex. $C_i = [0, 1 - 1/i]$)

(5) $\{C_i \in \mathcal{C} \mid i \in I\} \subseteq \mathcal{C} \Rightarrow \bigcap_{i \in I} C_i \in \mathcal{C}$

closure

(6) $\bar{A} \stackrel{\text{def.}}{=} A \cup L(A) \stackrel{\text{thm}}{=} \bigcap_{A \subseteq C,\ C \text{ closed}} C$ (Thm. 3.2.12)

closed $\Leftrightarrow$ closure (7) $$\boxed{\begin{array}{l} A \in \mathcal{C} \iff A = \bar{A} \\ A \in \mathcal{T} \iff A = A^\circ \end{array}}$$ (Ex. 3.2.14)

open $\Leftrightarrow$ interior

limit pt. (8) $$\boxed{a \in L(A) \iff \overline{A - \{a\}} \ni a}$$

double closure (9) $$\boxed{\bar{\bar{A}} = \bar{A}}$$

closure & $\cup$ (10) $$\boxed{\begin{array}{l} \overline{A \cup B} = \bar{A} \cup \bar{B} \\ \overline{\bigcup_{i \in I} A_i} \supseteq \bigcup_{i \in I} \bar{A}_i \end{array}}$$ (Ex. 3.2.11)

closure & $\cap$ (11) $$\boxed{\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}}$$ (show why $\subsetneq$ can happen by an example)

adherent points (12) $$\boxed{\begin{array}{l} A(A) = L(A) \sqcup I(A) \\ \qquad = \bar{A} \end{array}}$$

disjoint union (meaning $L(A) \cup I(A)$ + $L(A) \cap I(A) = \emptyset$)

boundary

(13) $\boxed{\partial A = \partial A^c}$

(14) $\boxed{\partial A \in \mathcal{C}}$ (equivalently $\overline{\partial A} = \partial A$ by (7))

(15)
$$\boxed{\begin{array}{l} A \in \mathcal{C} \iff \partial A \subseteq A \\ A \in \mathcal{T} \iff \partial A \cap A = \emptyset \end{array}}$$

(Hence $\partial A \cap A^\circ = \emptyset$)

(16) $A - A^\circ \subseteq \partial A$

(17) $\bar{A} - A^\circ = \partial A$

(18)
$$\boxed{\begin{aligned} \bar{A} &= A^\circ \sqcup \partial A \\ &= A \cup \partial A \end{aligned}}$$

(19) $\boxed{A^\circ = \bar{A} - \partial A}$

limit pts. & sequences

(20) $\boxed{x \in L(A) \iff \exists (a_n)_{n\in\mathbb{N}} \text{ in } (A - \{x\}) \text{ s.t. } a_n \to x}$ (Thm. 3.2.5)

(21) $x \in \bar{A} \iff \exists (a_n)_{n\in\mathbb{N}} \text{ in } A \text{ s.t. } a_n \to x$

(a slight rephrasing of (20))

(22) $\boxed{A \in \mathcal{C} \iff \forall (a_n)_{n\in\mathbb{N}} \text{ Cauchy in } A \quad \lim_{n\to\infty} a_n = a \in A}$

(Cauchy is unnecessary actually, convergent is good enough)

supremum & closure (& limit pts)

(23) If A is bounded above, then (Ex. 3.2.4 (a))

$$\boxed{\sup A \in \bar{A}}$$

(24) $\boxed{L(A) \in \mathcal{C}}$ (Ex. 3.2.7)