ex. 1   10 women, each with 3 children.

Q: How many ways are there to select one mother & one of her children as mother-&-child of the year?

A: $\underset{\substack{\uparrow \\ \text{10 women} \\ \text{to select}}}{10} \cdot \underset{\substack{\uparrow \\ \text{3 children} \\ \text{to select}}}{3} = \boxed{30}$ by the counting principle.

ex. 2   College planning committee consists of

3 freshmen
4 sophmores
5 juniors
2 seniors

Q: How many ways to sellect a subcommittee of size 4 consisting of 1 each?

A: $3 \cdot 4 \cdot 5 \cdot 2 = \boxed{120}$

ex. 3 Q: How many diff. 7-place licence plates if the first 3 places are letters & the last 4 are numbers (bet. 0 & 9)?

A: $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 26^3 \cdot 10^4 = \boxed{175{,}760{,}000}$

~~78,624,000~~

or

$26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 \cdot 7$

$= \boxed{78{,}624{,}000}$ if no repetitions are allowed

ex. 4 How many diff. funcions

$$f : \{x_1, \dots, x_n\} \longrightarrow \{0, 1\}$$

i.e. satisfying $f(x_i) = 0$ or 1 for each i?

A: $\underbrace{2 \cdot 2 \cdots 2}_{n \text{ of these}} = \boxed{2^n}$

ex. 5 How many batting orders are poss. for a 9-player baseball team?

A: $9! = \boxed{362,880}$

ex 6 Class with 6 men, 4 women takes an exam, & no two students get same score.

(a) How many diff. rankings are poss?

A: $10! = \boxed{3,628,800}$

(b) If the men are ranked separately from the women?

A: $6!4! = \boxed{17,280}$

principle of counting

<u>ex. 7</u> 10 books:

4 math

3 chem

2 history

1 language

<u>Q:</u> How many ways to arrange in a bookcase?

<u>A:</u>

$$\underbrace{4!}_{\text{\# ways to arrange subjects}}\;\underbrace{4!}_{\text{\# ways to arrange math books}}\;\overbrace{3!}^{\text{\# ways to arrange chem books}}\;\underbrace{2!}_{\text{\# ways to arrange hist. books}}\;\underbrace{1!}_{\text{one way to arrange one language book}} = \boxed{6912}$$

<u>ex. 8</u> How many ways to rearrange the letters in the word PEPPER?

<u>A:</u> 6 letters: 3P's, 2E's, one R $\Rightarrow$

$$\binom{6}{3,2,1} = \frac{6!}{3!\,2!\,1!} = \boxed{60}$$

ex. 9 flags, 4 white
3 red
2 blue

hung in a line to form a signal.

Q: How many arrangements?

A: $\binom{9}{4,3,2} = \frac{9!}{4!\,3!\,2!} = \boxed{1260}$

ex. How many diff. committees of size 3 can be formed from a group of 20 people?

A: $\binom{20}{3} = \frac{20!}{3!\,17!} = \boxed{1140}$

ex. From a group of 5 women & 7 men, how many committees of 2 women & 3 men?

A: $\binom{5}{2}\binom{7}{3} = \boxed{350}$

principle of counting

(5) The number of vectors $(x_1, \dots, x_n) \in \mathbb{N}_0^n$ satisfying $\sum_{i=1}^{n} x_i \geq k$, where $x_i = 0, 1$ & $0 \leq k \leq n$ (if $k > n$, no sol'ns exist, & if $k < 0$, there is no diff. to $k = 0$).

is

$$\sum_{j=k}^{n} \binom{n}{j} = \binom{n}{k} + \binom{n}{k+1} + \cdots + \binom{n}{n}$$

$$= \sum_{j=0}^{n} \binom{n}{j} - \sum_{j=0}^{k-1} \binom{n}{j}$$

$$= 2^n - \sum_{j=0}^{k-1} \binom{n}{j}$$

(6) $$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots + \binom{n}{r}\binom{m}{0}$$

$$= \sum_{j=0}^{r} \binom{n}{j}\binom{m}{r-j}$$

(10) Analytic proof, part (e):

(a) $$k\binom{n}{k} = k \cdot \frac{n!}{k!(n-k)!}$$

$$= \frac{n!}{(k-1)!(n-k)!}$$

(b) $$(n-k+1)\binom{n}{k-1} = (n-(k-1)) \cdot \frac{n!}{(k-1)!(n-(k-1))!}$$

$$= \frac{n!}{(k-1)!(n-k)!}$$

(c) $$n\binom{n-1}{k-1} = n \cdot \frac{(n-1)!}{(k-1)!(n-\cancel{1}-(k-\cancel{1}))!}$$

$$= \frac{n!}{(k-1)!(n-k)!}$$

Thus,

$$\boxed{k\binom{n}{k} = (n-k+1)\binom{n}{k-1} = n\binom{n-1}{k-1}}$$

(11) (a) $$\sum_{k=1}^{n} k\binom{n}{k} = n\cdot 2^{n-1}$$

<u>pf:</u> Using (10)

$$\sum_{k=1}^{n} k\binom{n}{k} = \sum_{k=1}^{n} n\binom{n-1}{k-1}$$

$$= n\sum_{k=1}^{n}\binom{n-1}{k-1}$$

$$= n\sum_{k=0}^{n-1}\binom{n-1}{k}$$

$$= n(1+1)^{n-1}$$

$$= n\cdot 2^{n-1}$$