# HW 8

## Sec. 5.4

(26) $A = PDP^{-1}$; $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

plus $tr(AB) = tr(BA) \implies$

$$tr\,A = tr\,(\underbrace{P}_{A}\underbrace{DP^{-1}}_{B})$$

$$= tr\,(DP^{-1}P)$$

$$= tr\,D$$

$$= \sum_{i=1}^{n} \lambda_i$$

(34) DNE!

Sec. 5.5

(2) $A = \begin{pmatrix} 5 & -5 \\ 1 & 1 \end{pmatrix}$

$$\det(A - \lambda I) = \det\begin{pmatrix} 5-\lambda & -5 \\ 1 & 1-\lambda \end{pmatrix}$$

$$= (5-\lambda)(1-\lambda) + 5$$

$$= \lambda^2 - 6\lambda + 10$$

$$= 0$$

$$\Longrightarrow \boxed{\lambda = \frac{6 \pm \sqrt{36-40}}{2} = 3 \pm i}$$

Take $\lambda_1 = 3 - i$ first: $A - (3-i)I$

$$= \begin{pmatrix} 5-(3-i) & -5 \\ 1 & 1-(3-i) \end{pmatrix}$$

$$= \begin{pmatrix} 2+i & -5 \\ 1 & -2+i \end{pmatrix}$$

argh!

$\frac{1}{2+i}$ =

so

$$\begin{pmatrix}2+i & -5\\ 1 & -2+i\end{pmatrix} \xrightarrow{-\frac{1}{5}(2-i)I+II} \begin{pmatrix}2+i & -5\\ 0 & 0\end{pmatrix}$$

$$\xrightarrow{-\frac{1}{5}(2-i)I} \begin{pmatrix}1 & 2-i\\ 0 & 0\end{pmatrix}$$

Let $y=t$, then $x = -(2-i)t$

$= -2+i$

$$\Longrightarrow \begin{pmatrix}x\\ y\end{pmatrix} = t\begin{pmatrix}-2+i\\ 1\end{pmatrix}$$

$$\Longrightarrow \boxed{\vec{v}_1 = \begin{pmatrix}-2+i\\ 1\end{pmatrix}}$$

Now $\lambda_2 = 3+i$: $A-(3+i)I = \begin{pmatrix}5-(3+i) & -5\\ 1 & 1-(3+i)\end{pmatrix}$

$$= \begin{pmatrix}2-i & -5\\ 1 & -2-i\end{pmatrix}$$

$$\xrightarrow{-\frac{1}{5}(2+i)I+II} \begin{pmatrix}2-i & -5\\ 0 & 0\end{pmatrix}$$

$$\xrightarrow{\frac{1}{5}(2+i)I} \begin{pmatrix}1 & -2-i\\ 0 & 0\end{pmatrix}$$

Letting $y=t$,
$x=(2+i)t$

$$\Longrightarrow \boxed{\vec{v}_2 = \begin{pmatrix}2+i\\ 1\end{pmatrix}}$$

# Sec. 6.1

(14) $d(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\|$

$$= \sqrt{(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})} = \sqrt{\left(\begin{pmatrix}0\\-5\\2\end{pmatrix}-\begin{pmatrix}-4\\-1\\8\end{pmatrix}\right)\cdot\left(\begin{pmatrix}0\\-5\\2\end{pmatrix}-\begin{pmatrix}2\\-3\\3\end{pmatrix}\right)}$$

$$= \sqrt{\begin{pmatrix}4\\-4\\-6\end{pmatrix}\cdot\begin{pmatrix}4\\-4\\-6\end{pmatrix}}$$

$$= \sqrt{16+16+36}$$

$$= \sqrt{68}$$

(24) Start with LHS:

$$\|\vec{u}+\vec{v}\|^2+\|\vec{u}-\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) + (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$$

$$= \vec{u}\cdot\vec{u} + \cancel{2\vec{u}\cdot\vec{v}} + \vec{v}\cdot\vec{v} + \vec{u}\cdot\vec{u} - \cancel{2\vec{u}\cdot\vec{v}} + \vec{v}\cdot\vec{v}$$

$$= 2\vec{u}\cdot\vec{u} + 2\vec{v}\cdot\vec{v}$$

$$= 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$$

✓

## Sec. 6.2

(14) $\vec{y} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}, \ \vec{u} = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$

$$\Longrightarrow \vec{v} = R_{\pi/2}\vec{u} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 7 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \end{pmatrix}$$

- $\vec{y} = \vec{y}_{\parallel} + \vec{y}_{\perp}$

$$\stackrel{\text{def}}{=} \pi_{\vec{u}}(\vec{y}) + \pi_{\vec{v}}(\vec{y}) \qquad (\pi_{\vec{u}} \equiv \text{proj}_{\vec{u}})$$

← tiredness!

$$= \frac{\vec{y}\circ\vec{u}}{\|\vec{u}\|^2}\vec{u} + \frac{\vec{y}\circ\vec{v}}{\|\vec{v}\|^2}\vec{v}$$

$$= \frac{\begin{pmatrix} 2 \\ 6 \end{pmatrix}\circ\begin{pmatrix} 7 \\ 1 \end{pmatrix}}{\left\|\begin{pmatrix} 7 \\ 1 \end{pmatrix}\right\|^2}\begin{pmatrix} 7 \\ 1 \end{pmatrix} + \frac{\begin{pmatrix} 2 \\ 6 \end{pmatrix}\circ\begin{pmatrix} -1 \\ 7 \end{pmatrix}}{\left\|\begin{pmatrix} -1 \\ 7 \end{pmatrix}\right\|^2}\begin{pmatrix} -1 \\ 7 \end{pmatrix}$$

$$= \frac{14+6}{50}\begin{pmatrix} 7 \\ 1 \end{pmatrix} + \frac{40}{50}\begin{pmatrix} -1 \\ 7 \end{pmatrix}$$

$$= \boxed{\frac{2}{5}\begin{pmatrix} 7 \\ 1 \end{pmatrix} + \frac{4}{5}\begin{pmatrix} -1 \\ 7 \end{pmatrix}}$$

(16) Let $\vec{y}_\perp = \vec{y} - \pi_{\vec{u}}(\vec{y}) = \vec{y} - \frac{\vec{y}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u}$ & note

$$\vec{u}\cdot\vec{y}_\perp = \vec{u}\cdot\left(\vec{y} - \left(\frac{\vec{y}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u}\right)\right)$$

$$= \vec{u}\cdot\vec{y} - \left(\frac{\vec{y}\cdot\vec{u}}{\cancel{\|\vec{u}\|^2}}\cancel{\vec{u}\cdot\vec{u}}\right)$$

$$= \vec{u}\cdot\vec{y} - \vec{u}\cdot\vec{y}$$

$$= 0$$

So $\vec{y}_\perp$ is indeed orthogonal to $\vec{u}$. Moreover,

$\vec{y} = \vec{y}_\parallel + \vec{y}_\perp = \pi_{\vec{u}}(\vec{y}) + (\vec{y} - \pi_{\vec{u}}(\vec{y}))$, so

$$d(\vec{y}, \operatorname{span}(\vec{u})) = \|\vec{y}_\perp\|$$

$$= \left\|\vec{y} - \frac{\vec{y}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u}\right\|$$

$$= \left\|\begin{pmatrix}-3\\9\end{pmatrix} - \frac{\begin{pmatrix}-3\\9\end{pmatrix}\cdot\begin{pmatrix}1\\2\end{pmatrix}}{\left\|\begin{pmatrix}1\\2\end{pmatrix}\right\|^2}\begin{pmatrix}1\\2\end{pmatrix}\right\|$$

$$= \left\|\begin{pmatrix}-3\\9\end{pmatrix} - \frac{18-3}{5}\begin{pmatrix}1\\2\end{pmatrix}\right\|$$

$$= \left\|\begin{pmatrix}-3\\9\end{pmatrix} - \begin{pmatrix}3\\6\end{pmatrix}\right\| = \left\|\begin{pmatrix}-6\\3\end{pmatrix}\right\| = 36+9$$

$$= \boxed{45}$$

# Sec. 6.3

(12) $$\pi_{\omega}(\vec{y}) = \pi_{\text{span}(\vec{v}_1, \vec{v}_2)}(\vec{y})$$

$$= \pi_{\vec{v}_1}(\vec{y}) + \pi_{\vec{v}_2}(\vec{y})$$

$$= \frac{\vec{y}\cdot\vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 + \frac{\vec{y}\cdot\vec{v}_2}{\|\vec{v}_2\|}\vec{v}_2$$

$$= \frac{\begin{pmatrix}3\\-1\\1\\13\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\-1\\2\end{pmatrix}}{\left\|\begin{pmatrix}1\\-2\\-1\\2\end{pmatrix}\right\|^2}\begin{pmatrix}1\\-2\\-1\\2\end{pmatrix} + \frac{\begin{pmatrix}3\\-1\\1\\13\end{pmatrix}\cdot\begin{pmatrix}-4\\1\\0\\3\end{pmatrix}}{\left\|\begin{pmatrix}-4\\1\\0\\3\end{pmatrix}\right\|^2}\begin{pmatrix}-4\\1\\0\\3\end{pmatrix}$$

$$= \frac{3+2-1+26}{1+4+1+4}\begin{pmatrix}1\\-2\\-1\\2\end{pmatrix} + \frac{-12-1+39}{16+1+9}\begin{pmatrix}-4\\1\\0\\3\end{pmatrix}$$

$$= \frac{30}{10}\begin{pmatrix}1\\-2\\-1\\2\end{pmatrix} + \frac{26}{26}\begin{pmatrix}-4\\1\\0\\3\end{pmatrix}$$

$$= \begin{pmatrix}3-4\\-6+1\\-3+0\\6+3\end{pmatrix}$$

$$\boxed{= \begin{pmatrix}-1\\-5\\-3\\9\end{pmatrix}}$$

(16) and therefore, for the record,

- $\vec{y}_\perp = \vec{y} - \pi_\omega(\vec{y})$

$$= \begin{pmatrix} 3 \\ -1 \\ 1 \\ 13 \end{pmatrix} - \begin{pmatrix} -1 \\ -5 \\ -3 \\ 9 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 4 \\ 4 \\ 4 \end{pmatrix}$$

$$= 4\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

so

$$d = (\vec{y}, \omega) = \|\vec{y}_\perp\| = 4\sqrt{4} = \boxed{8}$$

## Sec. 6.7

(22) $f(t) = 5t-3,\ g(t) = t^3 - t^2 \implies$

$f$ & $g$ are orthog.: $f \perp g$ $\Updownarrow$

$$\langle f, g\rangle \overset{def}{=} \int_0^1 f(t)g(t)\,dt$$

$$= \int_0^1 (5t-3)(t^3-t^2)\,dt$$

$$= \int_0^1 5t^4 - 8t^3 + 3t^2\,dt = t^5 - 2t^4 + t^3\Big|_0^1$$

$$= 1-2+1 = \boxed{0}$$

(24) $g(t) = t^3 - t^2 \implies$

$$\|g\| = \sqrt{\langle g, g\rangle}$$

$$= \sqrt{\int_0^1 g(t)^2 \, dt}$$

$$= \sqrt{\int_0^1 (t^3 - t^2)^2 \, dt}$$

$$= \sqrt{\int_0^1 t^6 - 2t^5 + t^4 \, dt}$$

$$= \frac{1}{7}t^7 - \frac{1}{3}t^6 + \frac{1}{5}t^5 \Big|_0^1$$

$$= \frac{1}{7} - \frac{1}{3} + \frac{1}{5}$$

$$= \frac{15 - 35 + 21}{105}$$

$$= \boxed{\frac{1}{105}}$$

## Sec. 7.1

(14) $A = \begin{pmatrix} 1 & -5 \\ -5 & 1 \end{pmatrix}$,

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & -5 \\ -5 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)^2 - 25$$

$$= \lambda^2 - 2\lambda - 24$$

$$= (\lambda - 6)(\lambda + 4)$$

$$= 0$$

$$\Longrightarrow \lambda_1 = -4$$

$$\lambda_2 = 6$$

For $\lambda_1 = -4$: $A + 4I = \begin{pmatrix} 5 & -5 \\ -5 & 5 \end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}}_{= \text{rref}}$

$$\Longrightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 6$: $A - 6I = \begin{pmatrix} -5 & -5 \\ -5 & -5 \end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}}_{= \text{rref}}$

$$\Longrightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Normalize each,

$$\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \quad \vec{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

& note $\vec{v}_1 \circ \vec{v}_2 = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} \circ \begin{pmatrix}1\\-1\end{pmatrix} = 0$, so $\vec{v}_1 \perp \vec{v}_2$,

and $\|\vec{v}_1\| = \frac{1}{\sqrt{2}}\left\|\begin{pmatrix}1\\1\end{pmatrix}\right\| = \frac{\sqrt{2}}{\sqrt{2}} = 1$ & similarly

$\|\vec{v}_2\| = 1$, so

$$\boxed{\beta = (\vec{v}_1, \vec{v}_2) = \left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right)}$$

is an orthonormal basis for $\mathbb{R}^2$ &

$$\boxed{P = M_{\beta\sigma} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} \in O(2)}$$

satisfies

$$P^{-1}AP = \frac{-1}{\sqrt{2}}\begin{pmatrix}-1 & -1\\-1 & 1\end{pmatrix}\begin{pmatrix}1 & -5\\-5 & 1\end{pmatrix}\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\right)$$

$$= \frac{1}{2}\underbrace{\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}1 & -5\\-5 & 1\end{pmatrix}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}-4 & -4\\6 & -6\end{pmatrix}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-8 & 0\\0 & 12\end{pmatrix} = \begin{pmatrix}-4 & 0\\0 & 6\end{pmatrix} = D$$

(16) $A = \begin{pmatrix} 6 & -2 \\ -2 & 9 \end{pmatrix}$

$$\det(A - \lambda I) = \det\begin{pmatrix} 6-\lambda & -2 \\ -2 & 9-\lambda \end{pmatrix}$$

$$= (6-\lambda)(9-\lambda) - 4$$

$$= \lambda^2 - 15\lambda + 50$$

$$= (\lambda - 5)(\lambda - 10)$$

$$= 0$$

$$\Longrightarrow \boxed{\lambda = 5, 10}$$

<u>Start w/ $\lambda_1 = 5$</u> $A - 5I = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$

$$\Longrightarrow \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

<u>$\lambda_2 = 10$:</u> $A - 10I = \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 1/2 \\ 0 & 0 \end{pmatrix}$

$$\Longrightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

<u>Normalize them:</u> $\boxed{\vec{v}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}}$

$$\Longrightarrow \boxed{P = M_{\beta\sigma} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \in O(2)}$$

& $P^{-1}AP = D = \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix}$