AW 7

Abstract Vector Spaces Notes

1.5 $\vec{x} = (1/n)_{n \in \mathbb{N}}$, $\vec{y} = (1/n^2)_{n \in \mathbb{N}}$

$$\Longrightarrow \boxed{\begin{aligned} 2\vec{x} - 3\vec{y} &= 2(1/n)_{n\in\mathbb{N}} - 3(1/n^2)_{n\in\mathbb{N}} \\ &= \left(\frac{2}{n} - \frac{3}{n^2}\right)_{n\in\mathbb{N}} \\ &= \left(\frac{2}{1} - \frac{3}{1^2}, \frac{2}{2} - \frac{3}{2^2}, \ldots\right) \end{aligned}}$$

and

$$2x_7 - 3y_7 = \frac{2}{7} - \frac{3}{7^2}$$

$$= \frac{14-3}{49} = \boxed{\frac{11}{14}}$$

<u>1.12</u> $f(x) = x^k|x| = x^k \cdot \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

$$= \begin{cases} x^{k+1}, & \text{if } x \geq 0 \\ -x^{k+1}, & \text{if } x < 0 \end{cases}$$

$$\Longrightarrow \quad f'(x) = \begin{cases} (k+1)x^k, & \text{if } x \geq 0 \\ -(k+1)x^k, & \text{if } x < 0 \end{cases}$$

contin. at 0! ($\lim_{x \to 0^\pm} f'(x) = 0$) $= f(0)$

$$\Longrightarrow \quad f''(x) = \begin{cases} (k+1)k\,x^{k-1}, & \text{if } x \geq 0 \\ -(k+1)k\,x^{k-1}, & \text{if } x < 0 \end{cases}$$

contin at $x = 0$

$\vdots$

$$\Longrightarrow \quad f^{(k)}(x) = \begin{cases} (k+1)!\,x, & \text{if } x \geq 0 \\ -(k+1)!\,x, & \text{if } x < 0 \end{cases}$$

contin. at $x = 0$!

<u>But</u> $f^{(k+1)}(x) = \begin{cases} (k+1)!, & \text{if } x > 0 \\ -(k+1)!, & \text{if } x < 0 \end{cases}$ <u>not</u> contin. at 0

Thus, $f \in C^k(\mathbb{R})$ but $f \notin C^{k+1}(\mathbb{R})$.

<u>1.29</u> Let $a \in F$ ($= \mathbb{R}$ or $\mathbb{C}$), & let $\vec{0} \in V$ be the zero vector. Then,

$$\cancel{a\vec{0}} + a\vec{0} \overset{(7)}{=} a(\vec{0}+\vec{0})$$

$$\overset{(3)}{=} a\vec{0}$$

$$\overset{(3)}{=} \cancel{a\vec{0}} + \vec{0}$$

cancellation law

$$\Longrightarrow \quad a\vec{0} = \vec{0}.$$

<u>5.10</u> $\mathcal{L}(V,W)$ is a vector space under pointwise addition,

$$S, T \in \mathcal{L}(V,W)$$

$\Rightarrow S+T : V \to W$ defined <u>pointwise</u> (Def 4.1)

$$(S+T)(v) \overset{\text{def}}{=} S(v) + T(v)$$

Well, since the y-values of $S+T$ lie in a vector space, W, the 8 vector space properties, e.g.

sum occurs in W

$$(1)\quad (S+T)(v) \overset{\text{def}}{=} S(v)+T(v)$$

$$= T(v)+S(v) \quad \text{bec. } W \text{ is a vector space}$$

$$\overset{\text{def}}{=} (T+S)(v)$$

Since this is true for all $v \in V$,

$$S+T = T+S$$

The other 7 properties follow similarly, though tediously, so I'll leave it to you!

Sec. 4.1

(8) Let $V = \{p(x) \in \mathbb{P}_n \mid p(0) = 0\}$, and consider what such a p must look like. It has degree (at most) n,

$$p(x) = \sum_{i=0}^{n} a_i x^i, \quad a_i \in \mathbb{R}$$

$$= a_0 + a_1 x + \cdots + a_n x^n$$

and satisfies

$$\begin{aligned} 0 &= p(0) \\ &= a_0 + a_1 \cdot 0 + \cdots + a_n 0^n \\ &= a_0 \end{aligned}$$

Therefore, we see that

$$V \subseteq \operatorname{span}(x, x^2, \ldots, x^n)$$

while clearly $V \supseteq \operatorname{span}(x, x^2, \ldots, x^n)$.

Therefore,

$$V = \text{span}(x, x^2, \ldots, x^n)$$

is an n-dimensional subspace of the $(n+1)$-dim. vector space $\mathbb{P}_n = \text{span}(1, x, x^2, \ldots, x^n)$.

It is also possible to use the def. of a subspace on V:

(1) V is nonempty, as the zero polynomial $p_0 \equiv 0 + 0x + \cdots + 0x^n$ clearly satisfies $p_0(0) = 0$.

(2) If $p, q \in V$, then $(p+q)(0) \stackrel{\text{def}}{=} p(0) + q(0)$
$$= 0 + 0$$
$$= 0$$

(3) If $p \in V$, $c \in \mathbb{R}$, then $(cp)(0) \stackrel{\text{def}}{=} cp(0)$
$$= c0$$
$$= 0.$$

(20 b) Let $V = \{ f \in C[a,b] \mid f(a) = f(b) \}$.

(1) The constant function $f_0 \equiv 0$ lies in V: it is contin. since for all $x_0 \in [a,b]$

$$\lim_{x \to x_0} f_0(x) = \lim_{x \to x_0} 0 = 0 = f_0(x_0)$$

& satisfies

$$f_0(a) = 0 = f_0(b)$$

(2) If $f, g \in V$, then $f(a) = f(b)$ & $g(a) = g(b)$

so

$$\begin{aligned}(f+g)(a) &\overset{\text{def}}{=} f(a) + g(a) \\ &\overset{\text{hyp.}}{=} f(b) + g(b) \\ &\overset{\text{def}}{=} (f+g)(b)\end{aligned}$$

~~(3)~~ And since by theorems (continuity & limit theorems) in Calc. 1, proved in undergrad real analysis, $f, g \in C[a,b] \Longrightarrow$ $f+g \in C[a,b]$, we see that $f+g \in V$.

(3) $f \in V$ & $c \in \mathbb{R} \Longrightarrow f(a) = f(b)$ & f is contin. on $[a,b] \Longrightarrow cf \in C[a,b]$ by Calc. I theorems, &

$$\begin{aligned}(cf)(a) &\overset{\text{def}}{=} c\,f(a)\\ &\overset{\text{hyp}}{=} c\,f(b)\\ &\overset{\text{def}}{=} (cf)(b)\end{aligned}$$

$\Longrightarrow cf \in V$.

Thus, V is closed under $+$ & $\cdot$, & contains 0, so it is a subspace.

Sec. 4.3

(38) Let $\beta = (1, \cos t, \ldots, \cos^6 t)$, and suppose $a_0, \ldots, a_6 \in \mathbb{R}$ satisfy

$$\sum_{i=0}^{6} a_i \cos^i t = 0$$

the zero function (= 0 for all $t \in \mathbb{R}$)

Then by cleverly choosing 7 different values of t, e.g.

$$t_0 = \pi/2$$
$$t_1 = 0$$
$$t_2 = \pi$$
$$t_3 = \pi/3$$
$$t_4 = 2\pi/3$$
$$t_5 = \pi/4$$
$$t_6 = \pi/6$$

we get 7 equations in the 7 unknowns $a_0, \dots, a_6$:

variables now

$$\sum_{i=0}^{6} a_i \overbrace{\cos^i(t_0)}^{\text{real \#/const./coeff.}} = 0 \quad \text{in } \mathbb{R} \text{ here}$$

$\vdots$

$$\sum_{i=0}^{6} a_i \cos^i(t_6) = 0$$

or

$$\begin{array}{l} \cos^i(t_0) \longrightarrow \\ \cos^i(t_1) \longrightarrow \\ \\ \\ \cos^i(t_6) \longrightarrow \end{array} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{\sqrt{3}}{2} & \frac{3}{4} & \cdots & \left(\frac{\sqrt{3}}{2}\right)^6 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

I computed this matrix explicitly and asked Matlab ~~Wolfram~~ to row-reduce it: it reduced to the identity. Try it! Therefore, $N(A) = \{\vec{0}\} \Rightarrow$ $a_0 = \cdots = a_6 = 0 \Rightarrow$ the functions in $\mathcal{B}$ are lin. indep.

Sec. 4.4 (14) Let $\beta = \{1-t^2;\ t-t^2;\ 2-2t+t^2\} \subseteq \mathbb{P}_2$

We are told that β is a basis, but let's make sure: if β is linearly indep., it is a basis since $\underbrace{|\beta|}_{\text{size}} = 3$ & $\dim \mathbb{P}_2 = 3$, so we just need to check that: suppose

$$a(1-t^2) + b(t-t^2) + c(2-2t+t^2) = 0$$

$\leftarrow$ the zero polynomial, identically 0

$$\Longrightarrow \quad (a+2c)\cdot 1 + (b-2c)t + (-a-b+c)t^2 = 0$$

Now, $\sigma = (1, t, t^2)$ is the standard basis for $\mathbb{P}_2$, so is linearly indep., which means

$$\begin{aligned} a + 2c &= 0 \\ b - 2c &= 0 \\ -a - b + c &= 0 \end{aligned}$$

$$\Longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ -1 & -1 & 1 & 0 \end{array}\right) \xrightarrow{I + III} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{II + III} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{3} III} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[2III + II]{-2III + I} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$\Longrightarrow$ $a = b = c = 0$, & β is lin. indep., so a basis for $\mathbb{P}_2$. OK, now to

$$p(t) = 3 + t - 6t^2$$

Since

$$[p]_\sigma = \begin{pmatrix} 3 \\ 1 \\ -6 \end{pmatrix} \quad \& \quad \begin{aligned} b_1(t) &= 1 - t^2 \\ b_2(t) &= t - t^2 \\ b_3(t) &= 2 - 2t + t^2 \end{aligned}$$

$$\Longrightarrow M_{\beta\sigma} = \begin{pmatrix} | & | & | \\ [b_1]_\sigma & [b_2]_\sigma & [b_3]_\sigma \\ | & | & | \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ -1 & -1 & 1 \end{pmatrix}$$

We have

$$[p]_\beta = M_{\sigma\beta} [p]_\sigma$$

$$= M_{\beta\sigma}^{-1} [p]_\sigma$$

used Wolfram

$$= \begin{pmatrix} -1 & -2 & -2 \\ 2 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ -6 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ -3 \\ -2 \end{pmatrix}$$

$$\Longrightarrow \boxed{p = 7b_1 - 3b_2 - 2b_2}$$

Let's verify: $7b_1(t) - 3b_2(t) - 2b_2(t)$

$$= 7(1-t^2) - 3(t - t^2) - 2(2 - 2t + t^2)$$

$$= (7-4)\cdot 1 + (-3+4)t + (-7+3-2)t^2$$

$$= 3 + t - 6t^2 = p(t) \quad \checkmark$$

(28) Let $p_1(t) = 1 - 2t^2 - t^3$

$p_2(t) = t + 2t^3$

$p_3(t) = 1 + t - 2t^2$

be in $\mathbb{P}_3 \cong \mathbb{R}^4$, & let $\sigma = (1, t, t^2, t^3)$ be the standard basis in $\mathbb{P}_3$, so that

$$[p_1]_\sigma = \begin{pmatrix} 1 \\ 0 \\ -2 \\ -1 \end{pmatrix}$$

$$[p_2]_\sigma = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}$$

$$[p_3]_\sigma = \begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix}$$

Since

$$\varphi_\sigma : \mathbb{P}_3 \to \mathbb{R}^4, \quad p \longmapsto \varphi_\sigma(p) = [p]_\sigma$$

is a linear isomorphism (Thm. 5.26, <u>Abstract Vector Spaces</u>), it preserves linear independence (Thm. 5.15, "), so

p_1, p_2, p_3 are linearly indep. in $\mathbb{P}_3$

$\iff [p_1]_\sigma, [p_2]_\sigma, [p_3]_\sigma$ are lin. indep. in $\mathbb{R}^4$

$\iff \begin{pmatrix}1\\0\\-2\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\2\end{pmatrix}, \begin{pmatrix}1\\1\\-2\\0\end{pmatrix}$ are lin. indep. in $\mathbb{R}^4$

$\iff a[p_1]_\sigma + b[p_2]_\sigma + c[p_3]_\sigma = \vec{0}$

$\implies a = b = c = 0$

$\iff \begin{pmatrix} | & | & | \\ [p_1]_\sigma & [p_2]_\sigma & [p_3]_\sigma \\ | & | & | \end{pmatrix} \begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$

$\implies \begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$\iff N(A) = \{\vec{0}\}, \quad A = \begin{pmatrix} | & | & | \\ [p_1]_\sigma & [p_2]_\sigma & [p_3]_\sigma \\ | & | & | \end{pmatrix}$

Well,

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -2 & 0 & -2 \\ -1 & 2 & 0 \end{pmatrix} \xrightarrow[\mathrm{I}+\mathrm{IV}]{2\mathrm{I}+\mathrm{III}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \xrightarrow[\text{then } -\mathrm{IV}+\mathrm{II}]{\frac{1}{2}\mathrm{IV}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow[\text{then } \mathrm{II}\Leftrightarrow\mathrm{IV}]{\mathrm{II}\Leftrightarrow\mathrm{III}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{-\mathrm{III}+\mathrm{IV}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \longleftarrow \text{has rank } 3$$

$\implies \dim N(A) = 0$

so $N(A) = \{\vec{0}\}$ & $a=b=0$, making p_1, p_2, p_3 lin. indep.

Sec. 4.5

(34) (a) $\beta = (1, \cos t, \cos^2 t, \cos^3 t, \cos^4 t, \cos^5 t, \cos^6 t)$

$\gamma = (1, \cos t, \cos 2t, \cos 3t, \cos 4t, \cos 5t, \cos 6t)$

and

$$\cos 2t = -1 + 2\cos^2 t$$
$$\cos 3t = -3\cos t + 4\cos^3 t$$
$$\cos 4t = 1 - 8\cos^2 t + 8\cos^4 t$$
$$\cos 5t = 5\cos t - 20\cos^3 t + 16\cos^5 t$$
$$\cos 6t = -1 + 18\cos^2 t - 48\cos^4 t + 32\cos^6 t$$

which ~~tell~~ tell us that, along w/ $1=1$ & $\cos t = \cos t$,

$$M_{\gamma\beta} = \begin{pmatrix} | & & | \\ [c_0]_\beta & \cdots & [c_6]_\beta \\ | & & | \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -3 & 0 & 5 & 0 \\ 0 & 0 & 2 & 0 & -8 & 0 & 18 \\ 0 & 0 & 0 & 4 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & -48 \\ 0 & 0 & 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 32 \end{pmatrix}$$

and, being upper triangular, satisfies

$$\det M_{\gamma\beta} = 1\cdot 1\cdot 2\cdot 4\cdot 8\cdot 16\cdot 32 \neq 0$$

$\Longrightarrow$ $M_{\gamma\beta} \in GL(7, \mathbb{R})$

$\Longrightarrow$ the <u>columns of $M_{\gamma\beta}$</u> form a basis for $\mathbb{R}^7$,

so in particular <u>are linearly indep</u>

$\Longrightarrow$ $c_0 = 1$, $c_1 = \cos t$, $\vdots$, $c_6 = \cos 6t$ <u>are linearly indep.</u>

bec. $\varphi_\beta : H \to \mathbb{R}^7$ is an isom., whence so is $\varphi_\beta^{-1} : \mathbb{R}^7 \to H$

and isomorphisms preserve linear independence, & also <u>bases</u>!

(b) Since γ is also a basis, $\varphi_\gamma : H \to \mathbb{R}^7$ is an isomorphism, so

$$\begin{tikzcd} H \arrow[rr, "\varphi_\gamma \circ \varphi_\beta^{-1}"] \arrow[dr, "\varphi_\beta^{-1}"'] & & H \\ & \mathbb{R}^7 \arrow[ur, "\varphi_\gamma"'] & \end{tikzcd}$$

$\varphi_\gamma \circ \varphi_\beta^{-1} \in GL(H)$ (an isomorphism fr. H to H)

and therefore sends bases (β) to bases (γ) (Thm. 5.19, 'Abstract Vector Spaces' notes).

Sec. 4.7 (18) Let T be the linear transformation of indefinite integration on $\mathcal{H}$, $Tf(t) \stackrel{\text{def}}{=} \int f(t)\,dt$.

Write/represent

$$f(t) \stackrel{\text{def}}{=} 5\cos^3 t - 6\cos^4 t + 5\cos^5 t - 12\cos^6 t$$

in β:

$$[f]_\beta = 5\vec{e}_4 - 6\vec{e}_5 + 5\vec{e}_6 - 12\vec{e}_7$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 5 \\ -6 \\ 5 \\ -12 \end{pmatrix} \in \mathbb{R}^7$$

Then,

$$[f]_\gamma = M_{\beta\gamma}[f]_\beta$$

$$= \left(\begin{array}{ccc|ccc|c} 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -3 & 0 & 5 & 6 \\ 0 & 0 & 2 & 0 & -8 & 6 & 18 \\ \hline 0 & 0 & 0 & 4 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & -48 \\ 0 & 0 & 0 & 0 & 0 & 16 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 32 \end{array}\right) \begin{pmatrix} 0 \\ 0 \\ 0 \\ \hline 5 \\ -6 \\ 5 \\ \hline -12 \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \\ \\ \\ \\ \\ \\ \end{pmatrix}$$

Plugging into Wolfram we get

$$M_{\beta\gamma}^{-1} = M_{\gamma\beta} = \frac{1}{32}\left(\begin{array}{ccc|ccc|c} 32 & 0 & 16 & 0 & 12 & 0 & 10 \\ 0 & 32 & 0 & 24 & 0 & 20 & 0 \\ 0 & 0 & 16 & 0 & 16 & 0 & 15 \\ \hline 0 & 0 & 0 & 8 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$\gamma = T(\beta)$ for some $T \in \mathcal{L}(H)$ $\iff$ $[T]_{\beta\gamma} = \underline{\underline{M_{\beta\gamma} \in GL(7,\mathbb{R})}}$

$\iff$ $T = I$ on H (see "Bases & Coords" notes)