HW 6

'Bases, Coords. & Reps.' notes:

(6.11) Let $S, T \in GL(\mathbb{R}^n)$, so that S^{-1}, T^{-1} exist & are also in $GL(\mathbb{R}^n)$.

Since

$$S \circ S^{-1} = S^{-1} \circ S = I_n$$

we see that S is the inverse of S^{-1},

$$\boxed{(S^{-1})^{-1} = S}$$

Also, since

$$\begin{aligned}(S \circ T) \circ (T^{-1} \circ S^{-1}) &= S \circ (T \circ T^{-1}) \circ S^{-1} \\ &= S \circ I_n \circ S^{-1} \\ &= S \circ S^{-1} \\ &= I_n\end{aligned}$$

& similarly $(T^{-1} \circ S^{-1}) \circ (S \circ T) = I_n$, we conclude that $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$.

(7.6) Since $\sin(\theta+\varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi$

$$\cos(\theta+\varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$$

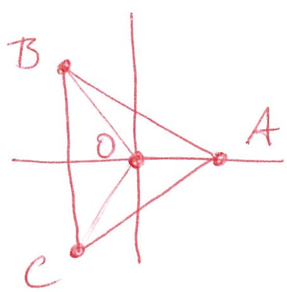
we have

$$R_\theta R_\varphi = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

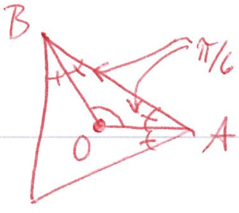
$$= \begin{pmatrix} \cos\theta\cos\varphi - \sin\theta\sin\varphi & -\cos\theta\sin\varphi - \sin\theta\cos\varphi \\ \sin\theta\cos\varphi + \cos\theta\sin\varphi & \cos\theta\cos\varphi - \sin\theta\sin\varphi \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta+\varphi) & -\sin(\theta+\varphi) \\ \sin(\theta+\varphi) & \cos(\theta+\varphi) \end{pmatrix}$$

$$= R_{\theta+\varphi}$$

(7.9) 

Since an equilateral triangle has $\pi/3$ internal angles, so



$$\measuredangle AOB = \pi - 2(\pi/6) = \pi - \pi/3 = 2\pi/3$$

Thus, to rotate vertex A to vertex B requires a $2\pi/3$ rotation

$$R_{2\pi/3}$$

Then

$$R_{4\pi/3} = R^2_{2\pi/3}$$

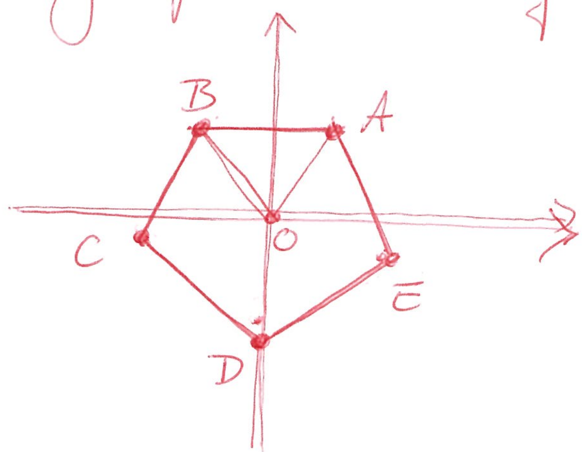
rotates A to C, B to A, C to B. This means

$$\boxed{R_3 = \{ I_2, R_{2\pi/3}, R_{4\pi/3} \}}$$

is the complete <u>rotation 3-group</u>, for if we compose $R_{4\pi/3}$ with another $R_{2\pi/3}$ we get

$$R_{2\pi/3} \circ R_{4\pi/3} = R_{\frac{2\pi}{3}+\frac{4\pi}{3}} = R_{2\pi} = I_2.$$

The pentagon proceeds similarly:



$\measuredangle AOB = \frac{2\pi}{5}$, so

$$\boxed{\mathcal{R}_5 = \{ I_2, R_{2\pi/5}, R_{4\pi/5} = R^2_{2\pi/5}, R_{6\pi/5} = R^3_{2\pi/5}, R_{8\pi/5} = R^4_{2\pi/5} \}}$$

(7.12) Let's compute:

$$(R_\theta\vec{x})\cdot(R_\theta\vec{x}) = \left(\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}\right)\cdot\left(\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}\right)$$

$$= \begin{pmatrix}x\cos\theta - y\sin\theta\\ x\sin\theta + y\cos\theta\end{pmatrix}\cdot\begin{pmatrix}x\cos\theta - y\sin\theta\\ x\sin\theta + y\cos\theta\end{pmatrix}$$

$$= (x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2$$

$$= x^2\cos^2\theta \cancel{- 2xy\sin\theta\cos\theta} + y^2\sin^2\theta +$$
$$x^2\sin^2\theta \cancel{+ 2xy\sin\theta\cos\theta} + y^2\cos^2\theta$$

$$= x^2(\sin^2\theta + \cos^2\theta) + y^2(\sin^2\theta + \cos^2\theta)$$

$$= x^2 + y^2$$

$$= \begin{pmatrix}x\\ y\end{pmatrix}\cdot\begin{pmatrix}x\\ y\end{pmatrix}$$

$$= \vec{x}\cdot\vec{x}$$

(7.15) Let $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2$. Since

$$\mathrm{proj}_{\vec{v}}\,\vec{x} = \frac{\vec{x}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \left(\frac{\vec{x}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$$

we have

$$\mathrm{proj}_{\vec{v}}\,\vec{e}_1 = \frac{\vec{e}_1\cdot\vec{v}}{\vec{v}\cdot\vec{v}} = \left(\frac{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\cdot\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}\right)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$= \frac{v_1}{v_1^2+v_2^2}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

and

$$\mathrm{proj}_{\vec{v}}\,\vec{e}_2 = \frac{\vec{e}_2\cdot\vec{v}}{\vec{v}\cdot\vec{v}} = \left(\frac{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\cdot\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}\right)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$= \frac{v_2}{v_1^2+v_2^2}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

we have

$$\boxed{\begin{aligned} [\mathrm{proj}_{\vec{v}}]_\sigma &\overset{\text{def}}{=} \begin{pmatrix} | & | \\ [\mathrm{proj}_{\vec{v}}\,\vec{e}_1]_\sigma & [\mathrm{proj}_{\vec{v}}\,\vec{e}_2]_\sigma \\ | & | \end{pmatrix} \\ &= \begin{pmatrix} \frac{v_1}{v_1^2+v_2^2}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} & \frac{v_2}{v_1^2+v_2^2}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \end{pmatrix} \\ &= \frac{1}{v_1^2+v_2^2}\begin{pmatrix} v_1^2 & v_1 v_2 \\ v_1 v_2 & v_2^2 \end{pmatrix} \end{aligned}}$$

(7.21) If $\ell_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, y = 2x \right\}$, then letting $x = 1$ gives $y = 2 \cdot 1 = 2$, so

$$\vec{c}_1 \overset{\text{def}}{=} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

is a basis vector for ℓ_1,

$$\ell_1 = \operatorname{span}(\vec{c}_1) = \operatorname{span}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

If we rotate $\vec{c}_1$ through $\pi/2$ we get an orthogonal vector

$$\vec{c}_2 \overset{\text{def}}{=} R_{\pi/2}\vec{c}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

and

$$\gamma = (\vec{c}_1, \vec{c}_2) = \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right)$$

is a basis for $\mathbb{R}^2$, in which

$$\left.\begin{aligned} R_{\ell_1}\vec{c}_1 &= \vec{c}_1 \\ R_{\ell_2}\vec{c}_2 &= -\vec{c}_2 \end{aligned}\right\} \Longrightarrow [R_{\ell_1}]_\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Therefore,

$$[R_{l_1}]_\sigma = M_{\gamma\sigma}[R_{l_1}]_\gamma M_{\gamma\sigma}^{-1}$$

$$= \begin{pmatrix} | & | \\ \vec{c}_1 & \vec{c}_2 \\ | & | \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} | & | \\ \vec{c}_1 & \vec{c}_2 \\ | & | \end{pmatrix}^{-1}$$

$$= \underbrace{\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}\underbrace{\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}^{-1}}$$

$$= \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\left(\frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\right)$$

$$= \boxed{\frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}}$$

Let's verify its correctness: we should have $[R_{l_1}]_\sigma[\vec{c}_1]_\sigma = \vec{c}_1$ & $[R_{l_1}]_\sigma[\vec{c}_2]_\sigma = -\vec{c}_2$:

$$[R_{l_1}]_\sigma[\vec{c}_1]_\sigma = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 5 \\ 10 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \vec{c}_1 \checkmark$$

$$[R_{l_1}]_\sigma[\vec{c}_2] = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 10 \\ -5 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -1 \end{pmatrix} = -\begin{pmatrix} -2 \\ 1 \end{pmatrix} = -\vec{c}_2 \checkmark$$

Since $[R_{\ell_2}]_\sigma = -\frac{1}{5}\begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix}$ by Ex. 7.18,

$$
\begin{aligned}
[R_{\ell_2} \circ R_{\ell_1}]_\sigma &= [R_{\ell_2}]_\sigma [R_{\ell_1}]_\sigma \\
&= \left(-\frac{1}{5}\begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix}\right)\left(\frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}\right) \\
&= \frac{-1}{25}\begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} \\
&= \frac{-1}{25}\begin{pmatrix} 0 & 25 \\ -25 & 0 \end{pmatrix} \\
&= \boxed{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = R_{\pi/2}}
\end{aligned}
$$

So $\boxed{\theta = \pi/2}$.

(7.22) Let R_{ℓ_1} & R_{ℓ_2} be two reflections.

Let

$\boxed{\vec{b}_1 = \begin{pmatrix} a \\ b \end{pmatrix}}$ be a <u>basis for ℓ_1</u>

$\boxed{\vec{c}_1 = \begin{pmatrix} c \\ d \end{pmatrix}}$ be a <u>basis for ℓ_2</u>

and rotate each by $\pi/2$,

$$\boxed{\vec{b}_2 = R_{\pi/2}\vec{b}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix}}$$

$$\boxed{\vec{c}_2 = R_{\pi/2}\vec{c}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} -d \\ c \end{pmatrix}}$$

to give bases

$$\boxed{\begin{aligned} \beta &= (\vec{b}_1, \vec{b}_2) = \left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} -b \\ a \end{pmatrix}\right) \\ \gamma &= (\vec{c}_1, \vec{c}_2) = \left(\begin{pmatrix} c \\ d \end{pmatrix}, \begin{pmatrix} -d \\ c \end{pmatrix}\right) \end{aligned}}$$

for $\mathbb{R}^2$ in which

$$\boxed{[R_{\ell_1}]_\beta = [R_{\ell_2}]_\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

since by design $R_{\ell_1}(\vec{b}_1) = \vec{b}_1$, $R_{\ell_1}\vec{b}_2 = -\vec{b}_2$, &

$R_{\ell_2}\vec{c}_1 = \vec{c}_1$, $R_{\ell_2}\vec{c}_2 = -\vec{c}_2$. Then,

$$[R_{\ell_1}]_\sigma = M_{\beta\sigma}[R_{\ell_1}]_\beta M_{\beta\sigma}^{-1}$$

$$= \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{-1}$$

$$= \frac{1}{a^2+b^2}\underbrace{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

$$= \frac{1}{a^2+b^2}\begin{pmatrix} a & b \\ b & -a \end{pmatrix}\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

$$= \frac{1}{a^2+b^2}\begin{pmatrix} a^2-b^2 & 2ab \\ 2ab & -(a^2-b^2) \end{pmatrix}$$

and similarly

$$[R_{\ell_2}]_\sigma = \frac{1}{c^2+d^2}\begin{pmatrix} c^2-d^2 & 2cd \\ 2cd & -(c^2-d^2) \end{pmatrix}$$

If we normalize $\vec{b}_1$ & $\vec{c}_1$ from the get-go, i.e. ensure $\|\vec{b}_1\| = \sqrt{a^2+b^2} = 1$ & $\|\vec{c}_1\| = \sqrt{c^2+d^2} = 1$, then $\|\vec{b}_2\| = \|\vec{c}_2\| = 1$ as well (we can <u>always</u> normalize a nonzero vector $\vec{x} \neq \vec{0}$, namely by scaling by $\frac{1}{\|\vec{b}_1\|}$, for $\left\|\frac{\vec{b}_1}{\|\vec{b}_1\|}\right\| = \cancel{\frac{1}{\|\vec{b}_1\|}\|\vec{b}_1\|} = 1$),

and

$$[R_{\ell_1}]_\sigma = \begin{pmatrix} a^2-b^2 & 2ab \\ 2ab & -(a^2-b^2) \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} A & B \\ B & -A \end{pmatrix}$$

$$[R_{\ell_2}]_\sigma = \begin{pmatrix} c^2-d^2 & 2cd \\ 2cd & -(c^2-d^2) \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} C & D \\ D & -C \end{pmatrix}$$

where

$$A = a^2-b^2 \qquad C = c^2-d^2$$
$$B = 2ab \qquad D = 2cd$$

Then,

$$[R_{\ell_2}\circ R_{\ell_1}]_\sigma = [R_{\ell_2}]_\sigma [R_{\ell_1}]_\sigma \qquad (*)$$

$$= \begin{pmatrix} C & D \\ D & -C \end{pmatrix}\begin{pmatrix} A & B \\ B & -A \end{pmatrix}$$

$$= \begin{pmatrix} AC+BD & -(AD-BC) \\ AD-BC & AC+BD \end{pmatrix}$$

$$= \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \begin{pmatrix} \text{here:} \\ \alpha = AC+BD \\ \beta = AD-BC \end{pmatrix}$$

The last thing we need to check is that $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in S^1$

unit circle

i.e. that $\alpha^2+\beta^2=1$, for then $\alpha = \cos\theta$
$\beta = \sin\theta$

for some $\theta = \cos^{-1}\alpha = \sin^{-1}\beta$. To see this, note that

$$[R_{\ell_1}]_\beta = [R_{\ell_2}]_\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Longrightarrow \det([R_{\ell_1}]_\sigma) = \det([R_{\ell_1}]_\beta)$$
$$= \det\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$= -1$$

and $\det([R_{\ell_2}]_\sigma) = \det([R_{\ell_2}]_\gamma)$
$$= \det\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$= -1$$

$$\Longrightarrow \alpha^2+\beta^2 = \det\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$
$$= \det([R_{\ell_2}\circ R_{\ell_1}]_\sigma)$$ by (*) above p. 12
$$= \det([R_{\ell_1}]_\sigma [R_{\ell_2}]_\sigma)$$
$$= \det([R_{\ell_1}]_\sigma)\det([R_{\ell_2}]_\sigma) = (-1)^2 = 1$$

Lastly, to find the angle θ, note that

$$R_\theta \vec{b}_1 = (R_{\ell_2} \circ R_{\ell_1})(\vec{b}_1) = R_{\ell_2}(R_{\ell_1}\vec{b}_1)$$

$$= R_{\ell_2}\vec{b}_1$$

$$\Longrightarrow \quad \vec{b}_1 \cdot (R_{\ell_2}\vec{b}_1) = \vec{b}_1 \cdot (R_\theta \vec{b}_1)$$

$$= \|\vec{b}_1\| \, \|R_\theta \vec{b}_1\| \cos\theta$$

$$= 1 \cdot 1 \cos\theta \quad \left(\begin{array}{c}\text{by}\\ \text{Exercise 7.12}\end{array}\right)$$

$$= \cos\theta$$

$$\Longrightarrow \quad \boxed{\theta = \cos^{-1}\left(\vec{b}_1 \cdot R_{\ell_2}\vec{b}_1\right)}$$

Since $R_{\ell_2}\vec{b}_1 = [R_{\ell_2}\vec{b}_1]_\sigma = [R_{\ell_2}]_\sigma [\vec{b}_1]_\sigma$

$$= \begin{pmatrix} c^2-d^2 & 2cd \\ 2cd & -(c^2-d^2) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

one can come up with a precise formula, but it won't be pretty, I imagine!

(7.24) Let ℓ be the line for R_ℓ, choose a basis vector $\vec{b}_1$ for ℓ, & let $\vec{b}_2 = R_{\pi/2}\vec{b}_1$, so that $\beta = (\vec{b}_1, \vec{b}_2)$ forms a basis for $\mathbb{R}^2$.

In this basis, any vector $\vec{x} \in \mathbb{R}^2$ may be expressed as

$$\vec{x} = a\vec{b}_1 + b\vec{b}_2$$

where, we note,

$$a\vec{b}_1 = \text{proj}_{\vec{b}_1}\vec{x} = \underbrace{\frac{\vec{x}\cdot\vec{b}_1}{\|\vec{b}_1\|^2}}_{=a}\vec{b}_1$$

$$b\vec{b}_2 = \text{proj}_{\vec{b}_2}\vec{x} = \underbrace{\frac{\vec{x}\cdot\vec{b}_2}{\|\vec{b}_2\|^2}}_{=b}\vec{b}_2$$

Anyway, from this we have

$$\boxed{\begin{aligned} R_\ell\vec{x} &= R_\ell(a\vec{b}_1 + b\vec{b}_2) \\ &= aR_\ell\vec{b}_1 + bR_\ell\vec{b}_2 \\ &= a\vec{b}_1 - b\vec{b}_2 \end{aligned}}$$

and therefore, since $\vec{b}_2 = R_{\pi/2}\vec{b}_1 \implies \vec{b}_1 \circ \vec{b}_2 = 0$,

$$(R_\ell \vec{x}) \circ (R_\ell \vec{x}) = (a\vec{b}_1 - b\vec{b}_2) \circ (a\vec{b}_1 - b\vec{b}_2)$$

$$= a^2(\vec{b}_1 \circ \vec{b}_1) - \underbrace{2ab(\vec{b}_1 \circ \vec{b}_2)}_{=0}$$

$$+ b^2(\vec{b}_2 \circ \vec{b}_2)$$

$$= a^2 \|\vec{b}_1\|^2 + b^2 \|\vec{b}_2\|^2$$

while

$$\vec{x} \circ \vec{x} = (a\vec{b}_1 + b\vec{b}_2) \circ (a\vec{b}_1 + b\vec{b}_2)$$

$$= a^2(\vec{b}_1 \circ \vec{b}_1) + \underbrace{2ab(\vec{b}_1 \circ \vec{b}_2)}_{=0} + b^2(\vec{b}_2 \circ \vec{b}_2)$$

$$= a^2 \|\vec{b}_1\|^2 + b^2 \|\vec{b}_2\|^2$$

so $(R_{\ell_1} \vec{x}) \circ (R_{\ell_1} \vec{x}) = \vec{x} \circ \vec{x}$.

Sec. 3.1

(12) $$\det\begin{pmatrix} 3 & 0 & 0 & 0 \\ 7 & -2 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 3 & -8 & 4 & -3 \end{pmatrix}$$

$$= 3\det\begin{pmatrix} -2 & 0 & 0 \\ 6 & 3 & 0 \\ -8 & 4 & 3 \end{pmatrix} - 0 + 0 - 0$$

$$= 3\left(-2\det\begin{pmatrix} 3 & 0 \\ 4 & 3 \end{pmatrix} - 0 + 0\right)$$

$$= 3(-2)\cdot 3^2$$

$$= \boxed{-54}$$

(37) $$A = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} \implies 5A = \begin{pmatrix} 15 & 5 \\ 20 & 10 \end{pmatrix}$$

$$\implies \boxed{\begin{array}{l} \det(5A) = 150 - 100 = 50 \\ \det A = \det\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} = 2 \end{array}}$$

while (placed beside the second line of the box)

Thus $\boxed{\det(5A) = 50 = 5^2\cdot 2 = 5^2 \det A.}$

$\underline{\underline{\text{not}}}$ $\det(5A) = 5\det A.$

(38) $\det(kA) = \det\left(k\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$

$$= \det\begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$

$$= k^2(ad - bc)$$

$$= k^2 \det A$$

(41) $A = \left|\det\begin{pmatrix} | & | \\ \vec{u} & \vec{v} \\ | & | \end{pmatrix}\right| = \left|\det\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}\right| = \boxed{6}$

Sec. 3.2

(6) Let $A = \begin{pmatrix} 3 & 3 & -3 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{pmatrix}$

$$\xrightarrow[-\frac{2}{3}\mathrm{I}+\mathrm{III}]{-\mathrm{I}+\mathrm{II}} \begin{pmatrix} 3 & 3 & -3 \\ 0 & 1 & -1 \\ 0 & -5 & -3 \end{pmatrix}$$

$$\xrightarrow{5\mathrm{II}+\mathrm{III}} \begin{pmatrix} 3 & 3 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & -8 \end{pmatrix}$$

$$\Longrightarrow \det A = \det\begin{pmatrix} 3 & 3 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & -8 \end{pmatrix} = 3 \cdot 1 \cdot (-8)$$

$$= \boxed{-24}$$

(22) $\det \begin{pmatrix} 5 & 1 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{pmatrix} \leftarrow A$

$$= 5 \det \begin{pmatrix} -3 & -2 \\ 5 & 3 \end{pmatrix} - 1 \det \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} + (-1) \det \begin{pmatrix} 1 & -3 \\ 0 & 5 \end{pmatrix}$$

$$= \cancel{5(-9+10)} - 3 \cancel{-5}$$

$$= \underline{\underline{-3}}$$

$\Longrightarrow$ $\underline{A \text{ is invertible}}$

(31) $A \in GL(n, \mathbb{R}) \Longrightarrow AA^{-1} = I$

$$\Longrightarrow 1 = \det I$$

$$= \det(AA^{-1})$$

$$= (\det A)(\det A^{-1})$$

$$\Longrightarrow \det(A^{-1}) = \frac{1}{\det A}$$

(34) $\det(PAP^{-1}) = (\det P)(\det A)(\det(P^{-1}))$

$$= \cancel{(\det P)}(\det A)\cancel{\frac{1}{\det P}}$$

$$= \det A.$$

(38) $A = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 3 \\ -1 & -3 \end{pmatrix}$

$$\Rightarrow AB = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ -1 & -3 \end{pmatrix} = \begin{pmatrix} 6 & -9 \\ -2 & 3 \end{pmatrix}$$

$$\Rightarrow \det A = 0$$
$$\det B = -9$$
$$\det(AB) = 0$$

$$\Rightarrow \det(AB) = 0 = 0(-9) = (\det A)(\det B).$$