Hw 5

Bases, Coords. & Reps.

(1.9) Since U & V are subspaces of $\mathbb{R}^n$, they each contain $\vec{0}$ & are closed under $+$ & $\cdot$, so

(i) $\vec{0} \in U$ & $\vec{0} \in V \implies \vec{0} \in U \cap V$

(ii) $\vec{x}, \vec{y} \in U \cap V \implies \vec{x}, \vec{y} \in U$ & $\vec{x}, \vec{y} \in V$

$\implies \vec{x} + \vec{y} \in U$ & $\vec{x} + \vec{y} \in V$

$\implies \vec{x} + \vec{y} \in U \cap V$

(iii) $\vec{x} \in U \cap V$, $c \in \mathbb{R} \implies \vec{x} \in U$ & $\vec{x} \in V$

$\implies c\vec{x} \in U$ & $c\vec{x} \in V$

$\implies c\vec{x} \in U \cap V$

Therefore $U \cap V$ is a subspace of $\mathbb{R}^n$.

Similarly, (i) $\vec{0} \in U,\ \vec{0} \in V \Longrightarrow \vec{0} = \underset{\substack{\cap \\ U}}{\vec{0}} + \underset{\substack{\cap \\ V}}{\vec{0}} \in U+V$

(ii) $\vec{x} = \vec{u}_1 + \vec{v}_1 \in U+V$

$\vec{y} = \vec{u}_2 + \vec{v}_2 \in U+V$

$$\Longrightarrow \vec{x}+\vec{y} = (\vec{u}_1+\vec{v}_1) + (\vec{u}_2+\vec{v}_2)$$

$$= \underset{\substack{\cap \\ U}}{(\vec{u}_1+\vec{u}_2)} + \underset{\substack{\cap \\ V}}{(v_1+v_2)} \quad \Big\} \text{ associativity}$$

bec. U & V are subspaces

$$\Longrightarrow \vec{x}+\vec{y} \in U+V$$

(iii) $\vec{x} = \vec{u}+\vec{v} \in U+V, \quad c \in \mathbb{R} \Longrightarrow$

$$c\vec{x} = c(\vec{u}+\vec{v}) = \underset{\substack{\cap \\ U}}{c\vec{u}} + \underset{\substack{\cap \\ V}}{c\vec{v}} \in U+V$$

(1.18) Let $\vec{u} \in N(A)$ & $\vec{v} \in Row(A)$. Then,

$$A\vec{u} = \vec{0} \quad \& \quad \vec{v} = a_1\vec{A}_1 + \cdots + a_m\vec{A}_m$$

So

$$\vec{v} \cdot \vec{u} = (a_1\vec{A}_1 + \cdots + a_m\vec{A}_m) \cdot \vec{u}$$

$$= a_1(\vec{A}_1 \cdot \vec{u}) + \cdots + a_m(\vec{A}_1 \cdot \vec{u})$$

$$= \begin{pmatrix} (\vec{A}_1 \cdot \vec{u}) \\ \vdots \\ (A_m \cdot \vec{u}) \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$$

dot prod. def. of $A\vec{u}$

$$= (A\vec{u}) \cdot \vec{a}$$

$$= \vec{0} \cdot \vec{a}$$

$$= 0$$

Since $\vec{u} \cdot \vec{v} = \vec{0}$ for all $\vec{u} \in N(A)$ & $\vec{v} \in Row(A)$, we must have $N(A) \cap Row(A) = \{\vec{0}\}$, for if $\vec{x} \in N(A) \cap Row(A)$, then $\vec{x} \in N(A)$ <u>and</u> $\vec{x} \in Row(A)$, so

$$\vec{x} \cdot \vec{x} = 0$$

but this means, if $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$,

$$x_1^2 + x_2^2 + \cdots + x_n^2 = \vec{x} \cdot \vec{x} = 0$$

and since each $x_i^2 \geq 0$, we must have

$$x_1 = x_2 = \cdots = x_n = 0$$

else, if any $x_i \neq 0$, $x_i^2 > 0$ & $\vec{x} \cdot \vec{x} > 0$. Thus,

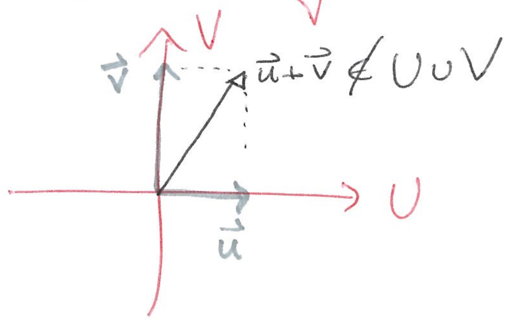
$$\vec{x} \in N(A) \cap Row(A) \implies \vec{x} = \vec{0}$$

$$\implies N(A) \cap Row(A) = \{\vec{0}\}.$$

and so $N(A) \oplus Row(A)$ is our sum (it's <u>direct</u>).

(1.20) Let $U = \operatorname{span}(\vec{e}_1) = x$-axis &

$V = \operatorname{span}(\vec{e}_2) = y$-axis



Here, $\vec{u} \in U$ & $\vec{v} \in V$, so $\vec{u}, \vec{v} \in U \cup V$, but $\vec{u}+\vec{v} \notin U \cup V$. This works only if $\vec{u}, \vec{v} \neq \vec{0}$, of course!

(1.25) Let $\vec{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 12 \\ 4 \end{pmatrix}$, $\vec{v}_4 = \begin{pmatrix} 70 \\ -22 \end{pmatrix}$.

If there are scalars $x_1, x_2, x_3, x_4 \in \mathbb{R}$ satisfying

$$\vec{0} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4 = \begin{pmatrix} | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \\ | & | & | & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = A\vec{x}$$

Then

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = A\vec{x} = \begin{pmatrix} -1 & 3 & 12 & 70 \\ 2 & 0 & 4 & -22 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Well,

$$\operatorname{rref}(A\,|\,\vec{0}) = \left(\begin{array}{cccc|c} 1 & 0 & 2 & -11 & 0 \\ 0 & 1 & 14/3 & 59/3 & 0 \end{array}\right)$$

so letting $x_3 = s$, $x_4 = t$ gives

$$x_1 = -2x_3 + 11x_4 = -2s + 11t$$

$$x_2 = -\frac{14}{3}x_3 - \frac{50}{3}x_4 = -\frac{14}{3}s - \frac{59}{3}t$$

so

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2s + 11t \\ -\frac{14}{3}s - \frac{50}{3}t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} -2 \\ -14/3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 11 \\ -50/3 \\ 0 \\ 1 \end{pmatrix}$$

We can take, e.g. $s = 3$, $t = 0$,

$$\vec{x} = \begin{pmatrix} 6 \\ -14 \\ 3 \\ 0 \end{pmatrix}$$

and note

$$6\vec{v}_1 - 14\vec{v}_2 + 3\vec{v}_3 + 0\vec{v}_4 = -6\begin{pmatrix} -1 \\ 2 \end{pmatrix} - 14\begin{pmatrix} 3 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 12 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 6 - 42 + 36 \\ -12 + 12 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \checkmark$$

(1.27) $\text{span}(\vec{v}_1, \dots, v_k) \overset{\text{def}}{=} \left\{ \sum_{i=1}^{k} a_i \vec{v}_i \;\middle|\; a_i \in \mathbb{R} \right\}$

While $\text{span}(\vec{v}_1) + \dots + \text{span}(\vec{v}_k) \overset{\text{def}}{=} \{ a_1\vec{v}_1 + \dots + a_k\vec{v}_k \mid a_i \in \mathbb{R} \}$

so the two are equal.

(1.28) Suppose $\text{span}(\vec{u}) \oplus \text{span}(\vec{v})$, i.e.

$$\text{span}(\vec{u}) \cap \text{span}(\vec{v}) = \{\vec{0}\}, \qquad (*)$$

and suppose we represent $\vec{0}$ in $\vec{u}$ & $\vec{v}$:

$$a\vec{u} + b\vec{v} = \vec{0}$$

We claim that if $(*)$ holds, $a = b = 0$:

well, $a\vec{u} + b\vec{v} = \vec{0} \implies a\vec{u} = -b\vec{v} \in \text{span}(\vec{u}) \cap \text{span}(\vec{v}) = \{\vec{0}\}$

$$\implies a\vec{u} = \vec{0}, \quad -b\vec{v} = \vec{0}$$

so, unless $\vec{u}$ or $\vec{v} = \vec{0}$ (which we disallow to avoid ~~trivialities~~), we must have $a = b = 0$, & $\vec{u}$ & $\vec{v}$ are lin. indep.

Conversely, suppose $\vec{u}$ & $\vec{v}$ are linearly independent, so that $a\vec{u}+b\vec{v}=\vec{0} \Longrightarrow a=b=0$, & consider any $\vec{x} \in \mathrm{span}(\vec{u}) \cap \mathrm{span}(\vec{v})$. We claim that $\vec{x}$ must be $\vec{0}$, so that $\mathrm{span}(\vec{u}) \cap \mathrm{span}(\vec{v})$
$= \{\vec{0}\}$

Well, if $\vec{x} \in \mathrm{span}(\vec{u}) \cap \mathrm{span}(\vec{v})$, then

$$\vec{x} = a\vec{u} = b\vec{v} \quad \text{for some } a, b \in \mathbb{R}$$

But then

$$a\vec{u} + (-b)\vec{v} = \vec{0}$$

so by their linear independence

$$a = -b = 0 \Rightarrow a = b = 0$$

& therefore $\vec{x} = a\vec{u} = b\vec{v} = \vec{0}$.

(3.12) Let $\gamma = \{\vec{c}_1, \vec{c}_2, \vec{c}_3, \vec{c}_4\} = \left\{ \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}, \begin{pmatrix}11\\0\\9\end{pmatrix} \right\}$

& $\beta = \{\vec{b}_1, \vec{b}_2\} = \left\{ \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\2\\0\end{pmatrix} \right\}$.

My thought is, since β is linearly indep., let's find one of the $\vec{c}_i \in \gamma$ for which

$$\beta \cup \{\vec{c}_i\}$$

spans $\mathbb{R}^3$, which would also make $\beta \cup \{\vec{c}_i\}$ a basis by Thm. 3.7 (notes), since a basis is a minimal spanning set. But in that case <u>all we need to check</u> is linear independence:

forgot to say: if $\beta \cup \{\vec{c}_i\}$ spans $\mathbb{R}^3$, then $\beta \cup \{\vec{c}_i, \vec{c}_j\}$ will span too, for any other $\vec{c}_j \in \gamma$.

Let's try $\vec{c}_1 = \begin{pmatrix}1\\2\\3\end{pmatrix}$: $\beta \cup \{\vec{c}_1\} = \left\{ \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix} \right\}$

Well, $a\begin{pmatrix}1\\0\\1\end{pmatrix} + b\begin{pmatrix}1\\2\\0\end{pmatrix} + c\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\Longrightarrow \begin{pmatrix}1&1&1\\0&2&2\\1&0&3\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Longrightarrow a=b=c=0$$

after row reduction. We could even use the determinant:

$$\det\begin{pmatrix}1&1&1\\0&2&2\\1&0&3\end{pmatrix} = 1\det\begin{pmatrix}2&2\\0&3\end{pmatrix} - 1\det\begin{pmatrix}0&2\\1&3\end{pmatrix} + \det\begin{pmatrix}0&2\\1&0\end{pmatrix}$$

$$= 6 - \cancel{(-2)} - \cancel{2}$$

$$= 6$$

$$\neq 0$$

$$\iff \begin{pmatrix}1&1&1\\0&2&2\\1&0&3\end{pmatrix} \in GL(3,\mathbb{R})$$

$\iff$ the cols. $\vec{b}_1, \vec{b}_2, \vec{c}_1$ are lin. indep.

OK, so $\beta \cup \{\vec{c}_1\}$ spans $\mathbb{R}^3$; therefore

$$\boxed{\beta \cup \{\vec{c}_1, \vec{c}_2\}}$$

will, too! $\gamma_1 = \{\vec{c}_3, \vec{c}_4\}$, $\gamma_2 = \{\vec{c}_1, \vec{c}_2\}$, and we've replaced γ_1 w/ β.

Sec. 4.1

(21) Yes it is, because it is a span:

$$H \overset{\text{def}}{=} \{ A \in M_2(\mathbb{R}) \mid A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \}$$

$$= \{ a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mid a, b, d \in \mathbb{R} \}$$

$$= \text{span}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

which is a subspace (cf. Example 1.7 in 'Bases, Coords, & Reps.' notes, proved in class).

(22) Fix $F \in M_{3,2}(\mathbb{R})$ and consider

$$H \overset{\text{def}}{=} \{ A \in M_{2,4}(\mathbb{R}) \mid FA = 0 \in M_{3,4}(\mathbb{R}) \}$$

Since $A = (\vec{a}_1\, \vec{a}_2\, \vec{a}_3\, \vec{a}_4)$ & $FA = (F\vec{a}_1\; F\vec{a}_2\; F\vec{a}_3\; F\vec{a}_4)$

$$= (\underbrace{\vec{0}}_{\vec{0} \in \mathbb{R}^3}\ \vec{0}\ \vec{0}\ \vec{0}) = 0$$

we see that $A = (\vec{a}_1\ \vec{a}_2\ \vec{a}_3\ \vec{a}_4) \in H \iff F\vec{a}_i = \vec{0}$ for each i

$\iff$ each $\vec{a}_i \in N(A)$,

it's pretty easy to see that H should be a subspace, because the columns of any $A \in H$ must lie in a subspace $N(A)$.

Anyway, formally:

(a) If $A, B \in H$, then $FA = FB = 0$

so $F(A+B) = FA + FB = 0 + 0 = 0$

$\implies A + B \in H$.

(b) $A \in H,\ c \in \mathbb{R} \implies F(cA)$

$= cFA = c0 = 0$

$\implies cA \in H$

(c) $0 \in H$, clearly, since

$F0 = F(\vec{0}\ \vec{0}\ \vec{0}\ \vec{0}) = (F\vec{0} \cdots F\vec{0})$

$= (\vec{0} \cdots \vec{0}) = 0$.

Thus, H is a subspace of $M_{2,4}(\mathbb{R})$.

Sec. 4.3

(14) B is in echelon form, which tells us that $\vec{b}_1, \vec{b}_3$ & $\vec{b}_5$ are pivot col's, & therefore lin. indep.. Consequently the columns $\underline{\vec{a}_1, \vec{a}_3 \,\&\, \vec{a}_5}$ form a basis for Col(A).

As to N(A), fr. B we parametrize the free variables,

$$x_2 = s$$
$$x_4 = t$$

& solve for

$$-9x_5 = 0 \implies x_5 = 0$$
$$5x_3 = 7x_4 - 8x_5 = 7t - 8\cdot 0 = 7t$$
$$x_1 = -2x_2 - 4x_4 - 5x_5 = -2s - 4t$$

$$\Longrightarrow \quad N(A) \ni \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2s-4t \\ s \\ 7t \\ t \\ 0 \end{pmatrix}$$

$$= s\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -4 \\ 0 \\ 7 \\ 1 \\ 0 \end{pmatrix}$$

basis vectors
for $N(A)$

(16) Label the vectors $\vec{v}_1$ through $\vec{v}_5$ & row reduce the matrix with the $\vec{v}_i$ as columns:

$$\operatorname{rref}(\vec{v}_1 \cdots \vec{v}_5) = \begin{pmatrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & -3 & 5 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which tells us $\vec{v}_1, \vec{v}_2$ & $\vec{v}_3$ are linearly indep. & form a basis for the set,

$$\boxed{\beta = (\vec{v}_1, \vec{v}_2, \vec{v}_3) = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 6 \\ -1 \\ 2 \\ -1 \end{pmatrix} \right)}$$

(20) $\vec{v}_1 - 3\vec{v}_2 + 5\vec{v}_3 = \vec{0} \implies \vec{v}_1 = 3\vec{v}_2 - 5\vec{v}_3$
$\in \text{Span}(\vec{v}_2, \vec{v}_3)$

whereas clearly

$\vec{v}_2 \neq c\vec{v}_3$ for any $c \in \mathbb{R}$

so $\vec{v}_2$ & $\vec{v}_3$ are lin. indep., & therefore form a basis for H!

Sec. 4.4

(2) $\beta = \left(\begin{pmatrix}3\\-5\end{pmatrix}, \begin{pmatrix}6\\7\end{pmatrix}\right) \implies M_{\beta\sigma} = \begin{pmatrix}3 & 6\\-5 & 7\end{pmatrix}$

$$\implies \boxed{\begin{aligned}\vec{x} = [\vec{x}]_\sigma &= M_{\beta\sigma}[\vec{x}]_\beta \\ &= \begin{pmatrix}3 & 6\\-5 & 7\end{pmatrix}\begin{pmatrix}8\\-5\end{pmatrix} \\ &= \begin{pmatrix}-6\\-75\end{pmatrix}\end{aligned}}$$

(4) $\beta = \left(\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ -7 \\ 3 \end{pmatrix} \right), \quad [\vec{x}]_\beta = \begin{pmatrix} -4 \\ 8 \\ -7 \end{pmatrix}$

$$\Longrightarrow \quad \vec{x} = [\vec{x}]_\sigma = M_{\beta\sigma} [\vec{x}]_\beta$$

$$= \begin{pmatrix} -1 & 3 & 4 \\ 2 & -5 & -7 \\ 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} -4 \\ 8 \\ -7 \end{pmatrix}$$

$$= \begin{pmatrix} 4 + 24 - 28 \\ -8 - 40 + 49 \\ 0 + 16 - 21 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix}}$$