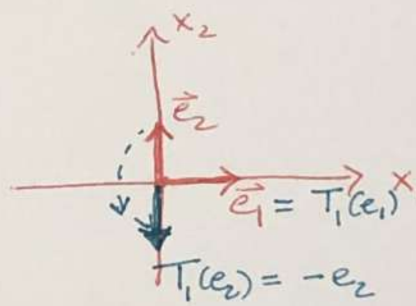
HW 4

Sec. 1.9

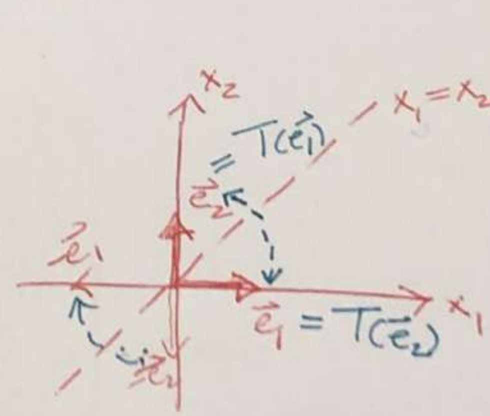
(8) I'm going to write the two operations separately, then compose them:

$T_1 : \mathbb{R}^2 \to \mathbb{R}^2$
reflection about x_1-axis



followed by T_2

$T_2 : \mathbb{R}^2 \to \mathbb{R}^2$
reflection about $x_1 = x_2$



Then, $T = T_2 \circ T_1$, & $T(\vec{e}_1) = T_2(T_1(\vec{e}_1)) = T_2(\vec{e}_1) =$

$T(\vec{e}_2) = T_2(T_1(\vec{e}_2)) = T_2(-\vec{e}_2) = -\vec{e}_1$

Therefore,

$$
\begin{aligned}
T(\vec{x}) &= T\begin{pmatrix} x \\ y \end{pmatrix} \\
&= T\left(x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \\
&= T(x\vec{e}_1 + y\vec{e}_2) \\
&= x\,T(\vec{e}_1) + y\,T(\vec{e}_2) \\
&= x\vec{e}_2 - y\vec{e}_1 \\
&= \begin{pmatrix} -y \\ x \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}
\end{aligned}
$$

$$\Longrightarrow \boxed{[T]_\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$$

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$$\cancel{\begin{pmatrix} x_1 - x_2 \\ -2x_1 + x_2 \\ x_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + x_2\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & -1 \\ -2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}}$$

ignore

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$$\cancel{\boxed{[T]_\sigma = \begin{pmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{pmatrix}}}$$

12. Since $[T]_\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos\pi/2 & -\sin\pi/2 \\ \sin\pi/2 & \cos\pi/2 \end{pmatrix}$

$$= R_{\pi/2}$$

we are done.

18. Write in terms of columns:

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_2 - 3x_1 \\ x_1 - 4x_2 \\ 0 \\ x_2 \end{pmatrix}$$

$$= x_1 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ -4 \\ 0 \\ 1 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} -3 & 2 \\ 1 & -4 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}}_{=[T]_\sigma} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Since $T(\vec{x}) = [T]_\sigma \vec{x}$, it's clear that T is linear.

Sec. 2.1

6. Only AB is defined, &

$$\left.\begin{aligned} AB &= \begin{pmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \vec{A}_1\cdot\vec{b}_1 & \vec{A}_1\cdot\vec{b}_2 \\ \vec{A}_2\cdot\vec{b}_1 & \vec{A}_2\cdot\vec{b}_2 \\ \vec{A}_3\cdot\vec{b}_1 & \vec{A}_3\cdot\vec{b}_2 \end{pmatrix} \\ &= \boxed{\begin{pmatrix} 0 & 14 \\ -3 & -9 \\ 13 & 4 \end{pmatrix}} \end{aligned}\right\} \text{ for part (b)}$$

or

$$\left.\begin{aligned} AB &= \left(A\vec{b}_1 \quad A\vec{b}_2\right) \\ &= \left(\left(b_{11}\vec{a}_1 + b_{21}\vec{a}_2\right) \quad \left(b_{12}\vec{a}_1 + b_{22}\vec{a}_2\right) \right) \\ &= \left(\left(1\begin{pmatrix} 4 \\ -3 \\ 3 \end{pmatrix} + 2\begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix}\right) \quad \left(3\begin{pmatrix} 4 \\ -3 \\ 3 \end{pmatrix} - 1\begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix}\right) \right) \\ &= \begin{pmatrix} 4-4 & 12+2 \\ -3+0 & -9+0 \\ 3+10 & 9-5 \end{pmatrix} = \boxed{\begin{pmatrix} 0 & 14 \\ -3 & -9 \\ 13 & 4 \end{pmatrix}} \end{aligned}\right\} \text{ fo pa (a}$$

10 $AB = \begin{pmatrix} 2 & -3 \\ -4 & 6 \end{pmatrix}\begin{pmatrix} 8 & 4 \\ 5 & 5 \end{pmatrix} = \begin{pmatrix} 1 & -7 \\ -2 & 14 \end{pmatrix}$

$AC = \begin{pmatrix} 2 & -3 \\ -4 & 6 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -7 \\ -2 & 14 \end{pmatrix}$ ✓

so yes, $AB = AC$, yet $B \neq C$.

Sec. 2.2

4 $\begin{pmatrix} 3 & -4 \\ 7 & -8 \end{pmatrix}^{-1} = \frac{1}{-24+28}\begin{pmatrix} -8 & 4 \\ -7 & 3 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -8 & 4 \\ -7 & 3 \end{pmatrix}$

$= \boxed{\begin{pmatrix} -2 & 1 \\ -7/4 & 3/4 \end{pmatrix}}$

14 If D is invertible, we can right-multiply by D^{-1} both sides of $(B-C)D = 0$:

$$0 = \underbrace{0D^{-1}}_{\text{why?}} = ((B-C)D)D^{-1}$$
$$= (B-C)(DD^{-1}) \quad \text{(associativity)}$$
$$= (B-C)I$$
$$= B-C \quad \text{(add } C \text{ to both sides)} \implies B = C.$$

(18) P invertible & $A = PBP^{-1}$

$\implies AP = PB$ (right multiplying both sides by P, & using associativity & $(PB)I = PB$)

$\implies P^{-1}AP = B$ (left multiplying both sides by P^{-1} + associativity & identity).

Sec. 2.3

(8) Invertible bec. $A\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$ (use back-substitution) $\Rightarrow N(A) = \{\vec{0}\}$,

& moreover A is square 4×4.

16 No, since $R(A) = \text{span}(\vec{a}_1, \ldots, \vec{a}_5) \subsetneq \mathbb{R}^5$;
which means A is <u>not onto</u>.

Or for any of the other equivalent reasons.