HW 3

Sec. 2.8

(6) Let $\vec{v}_1 = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -4 \\ -5 \\ 8 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} 8 \\ 2 \\ -9 \end{pmatrix}$.

Q: Is $\vec{w} \in \operatorname{span}(\vec{v}_1, \vec{v}_2)$?

A: $\vec{w} \in \operatorname{span}(\vec{v}_1, \vec{v}_2)$ iff $\vec{w} = a\vec{v}_1 + b\vec{v}_2$ for some $a, b \in \mathbb{R}$

$$= \begin{pmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$= A\vec{x}$$

where $A = \begin{pmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ 3 & -5 \\ -5 & 8 \end{pmatrix} \in M_{3,2}(\mathbb{R})$, $\vec{x} = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$

Well, $A\vec{x} = \vec{w}$ has a sol'n iff $(A|\vec{w})$ has the appropriate rref: but

$$\operatorname{rref}(A|\vec{w}) = \operatorname{rref}\left(\begin{array}{cc|c} 2 & -4 & 8 \\ 3 & -5 & 2 \\ -5 & 8 & -9 \end{array}\right)$$

$$= \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

all 0's $\longrightarrow$ nonzero,

so the answer is $\boxed{\text{No.}}$

8) Let $v_1 = \begin{pmatrix} -3 \\ 0 \\ 6 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 0 \\ -6 \\ 3 \end{pmatrix}$, $\vec{p} = \begin{pmatrix} 1 \\ 14 \\ -9 \end{pmatrix}$

Q: Is $\vec{p} \in Col(A) := span(\vec{v}_1, \vec{v}_2, \vec{v}_3)$, $A = \begin{pmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{pmatrix}$.

A: $\vec{p} \in Col(A) = span(\vec{v}_1, \vec{v}_2, \vec{v}_3) = R(A)$

iff $rref(A | \vec{p})$ is of the appropriate form:

Well,

$$rref(A|\vec{p}) = rref\left(\begin{array}{ccc|c} -3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 6 & -3 & 3 & -9 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ 0 & 1 & -3 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

which is of the right form, no nonzero element in column 4, row 3. In fact, if you let $B = rref(A|\vec{p})$, then

$$\vec{b}_4 = \begin{pmatrix} -5 \\ 7 \\ 0 \end{pmatrix} = -5\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 7\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -5\vec{b}_1 + 7\vec{b}_2$$

$$\Longrightarrow \quad \vec{p} = \vec{a}_4 = \begin{pmatrix} 1 \\ 14 \\ -9 \end{pmatrix} = -5\vec{a}_1 + 7\vec{a}_2 = -5\begin{pmatrix} -3 \\ 0 \\ 6 \end{pmatrix} + 7\begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}$$

from which we conclude that

$$\boxed{\vec{x} = \begin{pmatrix} -5 \\ 7 \\ 0 \end{pmatrix}}$$

is a soln. to $A\vec{x} = \vec{p}$. Thus,

$$\boxed{\text{Yes, } \vec{p} \in R(A) = Col(A)}$$

15. Let $\vec{u} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 10 \\ -3 \end{pmatrix}$.

Q: Does $\beta = (\vec{u}, \vec{v})$ form a basis?

A: β is a basis iff $\vec{u}, \vec{v}$ are linearly indep & span $\mathbb{R}^2$.

(1) $\vec{u}, \vec{v}$ are lin. indep. iff
"$a\vec{u} + b\vec{v} = \vec{0} \Rightarrow a = b = 0$"

but

$$\vec{0} = a\vec{u} + b\vec{v} = \begin{pmatrix} | & | \\ \vec{u} & \vec{v} \\ | & | \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$= A\vec{x}$$

where $A = \begin{pmatrix} | & | \\ \vec{u} & \vec{v} \\ | & | \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ -2 & -3 \end{pmatrix}$. But

$$\text{rref}(A \mid \vec{0}) = \text{rref}\left(\begin{matrix} 5 & 10 \\ -2 & -3 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \end{matrix}\right)$$

$$= \left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \end{matrix}\right)$$

so $a = b = 0$, & $\underline{\vec{u} \,\&\, \vec{v} \text{ are lin. indep.}}$

(2) spanning: $\vec{b} \in \text{span}(\vec{u}, \vec{v}) = \text{Col}(A)$

iff $A\vec{x} = \vec{b}$ has a sol'n. iff

$$\text{rref}(A \mid \vec{b}) = \text{rref}\left(\begin{matrix} 5 & 10 \\ -2 & -3 \end{matrix} \middle| \begin{matrix} b_1 \\ b_2 \end{matrix}\right)$$

$$= \left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \middle| \begin{matrix} h \\ k \end{matrix}\right)$$

where

$$\vec{x} = \begin{pmatrix} h = \\ k = \end{pmatrix} \begin{pmatrix} -\frac{3}{5} b_1 - 2b_2 \\ \frac{2}{5} b_1 + b_2 \end{pmatrix}$$

is the solution to $A\vec{x} = \vec{b}$. Thus, yes,

$\vec{u}, \vec{v}$ span $\mathbb{R}^2$.

$$\boxed{\text{Yes } \beta = (\vec{u}, \vec{v}) \text{ is a basis for } \mathbb{R}^2}$$

Sec. 2.9

3 $\beta = (\vec{b}_1, \vec{b}_2) = \left(\begin{pmatrix} 1 \\ -4 \end{pmatrix}, \begin{pmatrix} -2 \\ 7 \end{pmatrix} \right), \quad \vec{x} = \begin{pmatrix} -3 \\ 7 \end{pmatrix}$

$$\Longrightarrow [\vec{x}]_\beta = M_{\sigma\beta} [\vec{x}]_\sigma$$

$$= \begin{pmatrix} 1 & -2 \\ -4 & 7 \end{pmatrix}^{-1} \begin{pmatrix} -3 \\ 7 \end{pmatrix}$$

using our later methods

$$= \frac{1}{-1} \begin{pmatrix} 7 & 2 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ 7 \end{pmatrix}$$

$$= -\begin{pmatrix} -7 \\ -5 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

and indeed

$$(+7)\vec{b}_1 + 5\vec{b}_2 = 7 \begin{pmatrix} 1 \\ -4 \end{pmatrix} + 5 \begin{pmatrix} -2 \\ 7 \end{pmatrix}$$

$$= \begin{pmatrix} +7-10 \\ -28+35 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 7 \end{pmatrix}$$

$$= \vec{x} \quad \checkmark$$

This problem could also be done using row-reduction on $A\vec{u} = \vec{x}$, $A = \left(\vec{b}_1 \; \vec{b}_2 \right)$, $\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix}$.

5 $\vec{x} \in \mathrm{span}(\vec{b}_1, \vec{b}_2) \iff \vec{x} = a\vec{b}_1 + b\vec{b}_2$

$$= \begin{pmatrix} | & | \\ \vec{b}_1 & \vec{b}_2 \\ | & | \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\iff A\vec{u} = \vec{x}, \quad A = (\vec{b}_1, \vec{b}_2) = \begin{pmatrix} 1 & -3 \\ 5 & -7 \\ -3 & 5 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} 4 \\ 10 \\ -7 \end{pmatrix}$$

but

$$\mathrm{rref}(A \mid \vec{x}) = \mathrm{rref}\left(\begin{array}{cc|c} 1 & -3 & 4 \\ 5 & -7 & 10 \\ -3 & 5 & -7 \end{array}\right)$$

$$= \left(\begin{array}{cc|c} 1 & 0 & 1/4 \\ 0 & 1 & -5/4 \\ 0 & 0 & 0 \end{array}\right)$$

i.e. $a = 1/4$, $b = -5/4$.

Verify: $a\vec{b}_1 + b\vec{b}_2 = \frac{1}{4}\begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix} - \frac{5}{4}\begin{pmatrix} -3 \\ -7 \\ 5 \end{pmatrix}$

$$= \frac{1}{4}\begin{pmatrix} 1+15 \\ 5+35 \\ -3-25 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 16 \\ 40 \\ -28 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 10 \\ -7 \end{pmatrix} = \vec{x}$$

$$\Longrightarrow \boxed{[\vec{x}]_\beta = \begin{pmatrix} 1/4 \\ -5/4 \end{pmatrix}}$$

Sec. 1.8

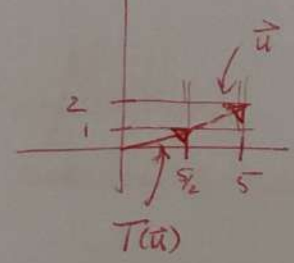
(8) $T_A : \mathbb{R}^4 \to \mathbb{R}^5 \implies A \in M_{5,4}(\mathbb{R})$

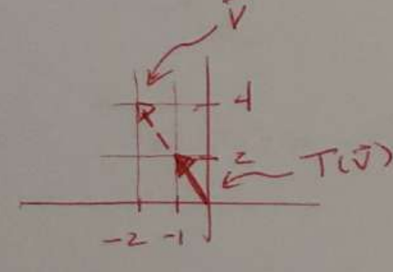
$\implies \boxed{\text{5 rows, 4 col's.}}$

(14) $T(\vec{u}) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 1 \end{pmatrix}$

$T(\vec{v}) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

This is a <u>dilation</u>, shrinking/compressing all vectors by a factor of 2:





(32) $T((-1)(x_1, x_2)) = T(-x_1, -x_2)$

$= (4(-x_1) - 2(-x_2),\ 3|-x_2$

$= (-4x_1 + 2x_2,\ 3|x_2|)$

$\overset{*}{\neq} T(-4x_1 + 2x_2,\ -3|x_2|)$

$= (-1)T(x_1, x_2)$