HW2, Linear Algebra

Sec 1.3

(14) Let $A = \begin{pmatrix} 1 & -2 & -6 \\ 0 & 3 & 7 \\ 1 & -2 & 5 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 11 \\ -5 \\ 9 \end{pmatrix}$. If there are scalars $a, b, c \in \mathbb{R}$ satisfying

$$a\vec{a}_1 + b\vec{a}_2 + c\vec{a}_3 = \vec{b} \quad (\vec{a}_i \text{ the cols. of } A)$$

then the scalars are the components of a solution vector $\vec{x}$, $\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, of the system

$$A\vec{x} = \vec{b}$$

So let's row reduce $(A \mid \vec{b}) = \left(\begin{array}{ccc|c} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{array}\right)$:

$$\operatorname{rref}(A \mid \vec{b}) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 245/33 \\ 0 & 1 & 0 & -41/33 \\ 0 & 0 & 1 & -2/11 \end{array}\right)$$

i.e. there is a unique sol'n:

$$\boxed{a = \frac{245}{3}, \quad b = \frac{-41}{33}, \quad c = \frac{-2}{11}}$$

Yes, $\vec{b}$ is a linear combo. of the $\vec{a}_i$.

(18) Let $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -3 \\ 1 \\ 8 \end{pmatrix}$, $\vec{y} = \begin{pmatrix} h \\ -5 \\ -3 \end{pmatrix}$.

Find $h \in \mathbb{R}$ such that $\vec{y} \in \operatorname{span}(\vec{v}_1, \vec{v}_2)$.

<u>Sol'n</u>: In order that $\vec{y} \in \operatorname{span}(\vec{v}_1, \vec{v}_2)$, we need scalars $a, b \in \mathbb{R}$ satisfying

$$\vec{y} = a\vec{v}_1 + b\vec{v}_2$$

i.e.

$$\begin{pmatrix} h \\ -5 \\ -3 \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + b\begin{pmatrix} -3 \\ 1 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -3 \\ 0 & 1 \\ -2 & 8 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

i.e.

$$A\vec{x} = \vec{b}, \quad \text{with } A = \begin{pmatrix} 1 & -3 \\ 0 & 1 \\ -2 & 8 \end{pmatrix} = \begin{pmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \vec{y} = \vec{b}.$$

Let's row-reduce $(A \mid \vec{b}) = (\vec{v}_1\ \vec{v}_2 \mid \vec{y})$ & find out:

$$(A|\vec{b}) = \left(\begin{array}{cc|c} 1 & -3 & h \\ 0 & 1 & -5 \\ -2 & 8 & -3 \end{array}\right) \xrightarrow{2\mathrm{I}+\mathrm{III}} \left(\begin{array}{cc|c} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 2 & 2h-3 \end{array}\right)$$

$$\xrightarrow{-2\mathrm{II}+\mathrm{III}} \left(\begin{array}{cc|c} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 0 & 2h+7 \end{array}\right)$$

$$\xrightarrow{3\mathrm{II}+\mathrm{I}} \left(\begin{array}{cc|c} 1 & 0 & h-15 \\ 0 & 1 & -5 \\ 0 & 0 & 2h+7 \end{array}\right) = \mathrm{rref}(A|\vec{b})$$

Thus, to have a solution at all, we need

$2h+7=0 \implies \boxed{h=-7/2}$, & in this

case the soln is $\boxed{\begin{array}{l} a = h-15 = -\frac{7}{2}-15 = \frac{-37}{2} \\ b = -5 \end{array}}$.

(29) We are given $\vec{v}_1 = \langle 5, -4, 3 \rangle, \quad m_1 = 2$

$$\vec{v}_2 = \langle 4, 3, -2 \rangle, \quad m_2 = 5$$

$$\vec{v}_3 = \langle -4, -3, -1 \rangle, \quad m_3 = 2$$

$$\vec{v}_4 = \langle -9, 8, 6 \rangle; \quad m_4 = 1$$

plugging these into the center of mass formula gives

$$\vec{V} \overset{\text{def}}{=} \frac{1}{m} \sum_{i=1}^{4} m_i \vec{v}_i$$

$$= \frac{1}{\sum_{i=1}^{4} m_i} \sum_{i=1}^{4} m_i \vec{v}_i$$

$$= \frac{1}{(2+5+2+1)} \Big(2\langle 5, -4, 3 \rangle + 5\langle 4, 3, -2 \rangle + 2\langle -4, -3, -1 \rangle + 1 \cdot \langle -9, 8, 6 \rangle \Big)$$

$$= \frac{1}{10} \langle 13, 9, 0 \rangle$$

$$= \boxed{\langle 13/10, 9/10, 0 \rangle}$$

(30) Yes, $\bar{\vec{v}} \in \text{span}(\vec{v}_1, \ldots, \vec{v}_k)$ by def.,

$$\bar{\vec{v}} = \frac{1}{m} \sum_{i=1}^{k} m_i \vec{v}_i$$

$$= \sum_{i=1}^{k} \left(\frac{m_i}{m}\right) \vec{v}_i \in \text{span}(\vec{v}_1, \ldots, \vec{v}_k)!$$

Sec. 1.4

(4) Let $A = \begin{pmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$,

& let $\vec{a}_1, \vec{a}_2, \vec{a}_3$ denote the columns of A & $\vec{A}_1, \vec{A}_2$ its rows. Then,

$$A\vec{b} = \begin{pmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1\vec{a}_1 + 1\vec{a}_2 + 1\vec{a}_3$$

$$= 1 \cdot \begin{pmatrix} 8 \\ 5 \end{pmatrix} + 1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} -4 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 8+3-4 \\ 5+1+2 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ 8 \end{pmatrix}$$

or, equivalently,

$$
\begin{aligned}
A\vec{b} &= \begin{pmatrix} - \vec{A}_1 - \\ - \vec{A}_2 - \end{pmatrix} \vec{b} \\
&= \begin{pmatrix} \vec{A}_1 \cdot \vec{b} \\ \vec{A}_2 \cdot \vec{b} \end{pmatrix} \\
&= \begin{pmatrix} \langle 8, 3, -4 \rangle \cdot \langle 1, 1, 1 \rangle \\ \langle 5, 1, 2 \rangle \cdot \langle 1, 1, 1 \rangle \end{pmatrix} \\
&= \begin{pmatrix} 8 \cdot 1 + 3 \cdot 1 - 4 \cdot 1 \\ 5 \cdot 1 + 1 \cdot 1 + 2 \cdot 1 \end{pmatrix} \\
&= \begin{pmatrix} 7 \\ 8 \end{pmatrix}
\end{aligned}
$$

(14) $\vec{u} = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$, $A = \begin{pmatrix} 5 & 8 & 7 \\ 0 & 1 & -1 \\ 1 & 3 & 0 \end{pmatrix}$. Let $\vec{a}_i$ denote the columns of A. Q: Is $\vec{u} \in \text{span}(\vec{a}_1, \vec{a}_2, \vec{a}_3)$?

A: I don't know until I rephrase the question in terms I can answer. But, this is easy, bec.

$$\vec{u} \in \text{span}(\vec{a}_1, \vec{a}_2, \vec{a}_3)$$

means

$$\vec{u} = a\vec{a}_1 + b\vec{a}_2 + c\vec{a}_3, \text{ for some } a, b, c \in \mathbb{R}$$

$$= \begin{pmatrix} | & | & | \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{by def. of matrix-vector prod.}$$

$$= A\vec{x}, \text{ where } \vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Ah, so rephrasing this means: "<u>Does $A\vec{x} = \vec{u}$ have a sol'n?</u>" I now know how to answer this: find rref$(A|\vec{u})$:

$$\text{rref}(A|\vec{u}) = \text{rref}\left(\begin{array}{ccc|c} 5 & 8 & 7 & 2 \\ 0 & 1 & -1 & -3 \\ 1 & 3 & 0 & 2 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

which is an <u>inconsistent system</u>, having <u>no soln</u>.

(18) Since $\text{rref}(B) = \begin{pmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, we see that B only has 3 pivot cols., $\vec{b}_1, \vec{b}_2, \vec{b}_4$, and that $\vec{b}_3 = -5\vec{b}_1 + \vec{b}_2 \in \text{span}(\vec{b}_1, \vec{b}_2, \vec{b}_3)$.

Indeed, try $\underline{\vec{y} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}}$, & you'll see that $\underline{B\vec{x} = \vec{y}}$ <u>is inconsistent</u>, which (by Thm. 4) means <u>it has no soln.</u>, & $\therefore$ the cols $\vec{b}_i$ do <u>not</u> span $\mathbb{R}^4$.

(22) Does $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ span $\mathbb{R}^3$, where $\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ -3 \\ 8 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 4 \\ -1 \\ -5 \end{pmatrix}$? By thm. 4, this will be true precisely when $B = \begin{pmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{pmatrix}$ has a pivot in each row, & $\therefore$ when $\operatorname{rref}(B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. But this is the case here, so <u>yes</u>!

(26) Since $3\vec{u} - 5\vec{v} - \vec{w} = \vec{0}$, $3\vec{u} - 5\vec{v} = \vec{w}$,

i.e. $\begin{pmatrix} | & | \\ \vec{u} & \vec{v} \\ | & | \end{pmatrix} \begin{pmatrix} 3 \\ -5 \end{pmatrix} = \vec{w}$

i.e. $\boxed{\begin{aligned} x_1 &= 3 \\ x_2 &= -5 \end{aligned}}$

Sec. 1.5

(10) $A = \begin{pmatrix} 1 & 3 & 0 & -4 \\ 2 & 6 & 0 & -8 \end{pmatrix}$, so

$$\mathrm{rref}(A \mid \vec{0}) = \left(\begin{array}{cccc|c} 1 & 3 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

pivot. free variables

Let

$$\left.\begin{array}{l} x_2 = s_1 \\ x_3 = s_2 \\ x_4 = s_3 \end{array}\right\} \Longrightarrow \begin{aligned} x_1 &= -3x_2 + 4x_4 \\ &= -3s_1 + 4s_3 \end{aligned}$$

$$\Longrightarrow \boxed{\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3s_1 + 4s_3 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = s_1 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + s_3 \begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \end{pmatrix}}$$

I.e. the solution set is the span of the 3 vectors above.

(26) $A\vec{x} = \vec{b}$ has a unique solution precisely (i.e. iff) $A\vec{x} = \vec{0}$ has only the trivial sol'n.

Pf: Suppose $A\vec{x} = \vec{b}$ has a unique sol'n (for each $\vec{b}$). Then, for $\vec{b} = \vec{0}$ it'll have a unique sol'n, & this solution will have to be $\vec{0}$, since this is possible always, & quite obviously: $\underbrace{A\vec{0}}_{\in \mathbb{R}^n} = 0\vec{a}_1 + \cdots + 0\vec{a}_n = \vec{0} \in \mathbb{R}^m$. This, by assumption, is unique!

Now suppose $A\vec{x} = \vec{0} \Longrightarrow \vec{x} = \vec{0}$ & consider what happens for general $\vec{b} \in \mathbb{R}^m$ in $A\vec{x} = \vec{b}$. We're told $A\vec{x} = \vec{b}$ has a sol'n, $\vec{x} \in \mathbb{R}^n$, & we must show $\vec{x}$ is unique: suppose, besides the one sol'n $\vec{x}$ we're told about there's another, $\vec{y} \in \mathbb{R}^n$.

I'm gonna show that $\vec{x} = \vec{y}$: we have, I suppose,

$$A\vec{x} = \vec{b} = A\vec{y}$$

but this means

$$\vec{0} = A\vec{x} - A\vec{y}$$
$$= A(\vec{x} - \vec{y})$$

& since this eq'n only has the trivial sol'n, we have

$$\vec{x} - \vec{y} = \vec{0}, \quad \& \therefore \vec{x} = \vec{y} \qquad \boxtimes$$

Sec. 1.7

(27) A matrix A, 7×5, must have 5 pivot col's to span $\mathbb{R}^5$

(28) $\boxed{5}$ by same reasoning as

(38) T (will discuss this in class.)