# HW1, Linear Algebra

## Sec. 1.1

(2)

$$\begin{aligned} \mathrm{I}&: 2x_1 + 4x_2 = -4 \\ \mathrm{II}&: 5x_1 + 7x_2 = 11 \end{aligned} \xrightarrow{-\frac{5}{2}\mathrm{I} + \mathrm{II}} \begin{aligned} 2x_1 + 4x_2 &= -4 \quad &: \mathrm{I} \\ -3x_2 &= 21 \quad &: \mathrm{II} \end{aligned}$$

$$\xrightarrow[\frac{-1}{3}\mathrm{II}]{\frac{1}{2}\mathrm{I},} \begin{aligned} x_1 + 2x_2 &= -2 \quad &: \mathrm{I} \\ x_2 &= -7 \quad &: \mathrm{II} \end{aligned}$$

$$\xrightarrow{-2\mathrm{II} + \mathrm{I}} \boxed{\begin{aligned} x_1 \quad &= 12 \\ x_2 &= -7 \end{aligned}} \begin{aligned} &: \mathrm{I} \\ &: \mathrm{II} \end{aligned}$$

i.e. $\boxed{P = (12, -7)}$

(4)

$$\begin{aligned} x_1 - 5x_2 &= 1 \quad \mathrm{I} \\ 3x_1 - 7x_2 &= 5 \quad \mathrm{II} \end{aligned} \xrightarrow{-3\mathrm{I} + \mathrm{II}} \begin{aligned} x_1 - 5x_2 &= 1 \\ 8x_2 &= 2 \end{aligned}$$

$$\xrightarrow{\frac{1}{8}\mathrm{II}} \begin{aligned} x_1 - 5x_2 &= 1 \\ x_2 &= 1/4 \end{aligned}$$

$$\xrightarrow{5\mathrm{II} + \mathrm{I}} \boxed{\begin{aligned} x_1 \quad &= 9/4 \\ x_2 &= 1/4 \end{aligned}}$$

i.e. $\boxed{P = (9/4, 1/4)}$

(8)
$$\left(\begin{array}{ccc|c} 1 & -4 & 9 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right) \xrightarrow{\frac{1}{2}\mathrm{III}} \left(\begin{array}{ccc|c} 1 & -4 & 9 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{-7\mathrm{III}+\mathrm{II}} \left(\begin{array}{ccc|c} 1 & -4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{4\mathrm{II}+\mathrm{I}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

i.e. $\boxed{x = y = z = 0}$

(12)
$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= -4 \\ 3x_1 - 7x_2 + 7x_3 &= -8 \\ -4x_1 + 6x_2 - x_3 &= 7 \end{aligned}$$

$$\Longrightarrow \left(\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 3 & -7 & 7 & -8 \\ -4 & 6 & -1 & 7 \end{array}\right) \xrightarrow[4\mathrm{I}+\mathrm{III}]{-3\mathrm{I}+\mathrm{II}} \left(\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & -6 & 15 & -9 \end{array}\right)$$

$$\xrightarrow{3\mathrm{II}+\mathrm{III}} \left(\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & 0 & 0 & 3 \end{array}\right)$$

inconsistent!

(20) $\left(\begin{array}{cc|c} 1 & h & -3 \\ -2 & 4 & 6 \end{array}\right) \xrightarrow{\frac{1}{2}\mathrm{II}} \left(\begin{array}{cc|c} 1 & h & -3 \\ -1 & 2 & 3 \end{array}\right)$

$\xrightarrow{\mathrm{I}+\mathrm{II}} \left(\begin{array}{cc|c} 1 & h & -3 \\ 0 & 2+h & 0 \end{array}\right)$

This means $x + hy = -3$
$(2+h)y = 0$

so there are 2 cases:

① $h = -2$: in this case $y$ is free, because our system becomes

$x - 2y = -3$

so letting $y = t$, $x = 2t - 3$

& $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2t-3 \\ t \end{pmatrix} = t\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}$

(infinitely many sol'ns).

② $h \neq -2$: in this case, our system reduces to $\left(\begin{array}{cc|c} 1 & h & -3 \\ 0 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & -3-h \\ 0 & 1 & 0 \end{array}\right)$

so $x = -3-h$, $y = 0$.

Sec. 1.2

(2) (a) rref

(b) not rref (col's. 2 & 3 have too many nonzero terms)

(c) not rref (" ")

(d) not rref (col's 3,4,5 have too many nonzero's)

---

pivot cols. of A are $\vec{a}_1, \vec{a}_2, \vec{a}_4$

O = pivot elements

(4) Let $A = \begin{pmatrix} \boxed{\begin{matrix}1\\3\\5\end{matrix}} & \boxed{\begin{matrix}3\\5\\7\end{matrix}} & \begin{matrix}5\\7\\9\end{matrix} & \boxed{\begin{matrix}7\\9\\1\end{matrix}} \end{pmatrix}$

$=\vec{a}_1$, col. 1 $\quad =\vec{a}_2$ col. 2 $\quad =\vec{a}_3$ col 3. $\quad =\vec{a}_4$ col$_4$

$$\xrightarrow[-5\mathrm{I}+\mathrm{III}]{-3\mathrm{I}+\mathrm{II}} \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{pmatrix}$$

$$\xrightarrow[-\frac{1}{2}\mathrm{III}]{-\frac{1}{4}\mathrm{II}} \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 4 & 8 & 17 \end{pmatrix}$$

$$\xrightarrow{-4\mathrm{II}+\mathrm{III}} \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{5}\mathrm{III}} \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[-7\mathrm{III}+\mathrm{I}]{-3\mathrm{III}+\mathrm{II}} \begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

pivot columns in rref(A)

$$\underbrace{\begin{pmatrix} \boxed{\begin{matrix}1\\0\\0\end{matrix}} & \boxed{\begin{matrix}0\\1\\0\end{matrix}} & \begin{matrix}-1\\2\\0\end{matrix} & \boxed{\begin{matrix}0\\0\\1\end{matrix}} \end{pmatrix}}_{=\mathrm{rref}(A)}$$

O = pivot element

$$(12)\quad \underbrace{\left(\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{array}\right)}_{\text{this is } (A|\vec{b})} \xrightarrow{I+III} \left(\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & 8 & 12 \end{array}\right)$$

$$\xrightarrow{4II+III} \left(\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \text{ this is rref}(A|\vec{b})$$

This means

$$x_1 - 7x_2 + 0x_3 + 6x_4 = 5$$

$$x_3 - 2x_4 = -3$$

free variables

dependent variables (in pivot positions)

Let's parametrize the free variables:

$$\boxed{x_2 = s, \quad x_4 = t}$$

then

$$\boxed{\begin{aligned} x_1 &= 7x_2 - 6x_4 + 5 = 7s - 6t + 5 \\ x_3 &= 2x_4 - 3 = 2t - 3 \end{aligned}}$$

Therefore, your typical solution vector is

$$\boxed{\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} &= \begin{pmatrix} 7s - 6t + 5 \\ s \\ 2t - 3 \\ t \end{pmatrix} \\ &= s\begin{pmatrix} 7 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -6 \\ 0 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ 0 \\ -3 \\ 0 \end{pmatrix} \\ &= s\vec{u} + t\vec{v} + \vec{s}_0 \\ &\in \operatorname{span}(\vec{u}, \vec{v}) + s_0 \end{aligned}}$$

where

$$\vec{u} = \begin{pmatrix} 7 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} -6 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \quad s_0 = \begin{pmatrix} 5 \\ 0 \\ -3 \\ 0 \end{pmatrix}$$

<u>Remark</u>: To find some particular solutions, you just take different values of $s$ & $t$, whatever you want, & add $s\vec{u} + t\vec{v} + s_0$. For ex., taking $s = t = 0$ gives $s_0 = \langle 5, 0, -3, 0 \rangle$ as a sol'n. $\boxtimes$

(22) (a) T (a thm., to be proven)

(b) F (if (a) is true, (b) must be false)

(c) T (by def.)

(d) T (it contains infinitely many sol'ns.)

(e) T (terminology, but also our avowed goal!)