Linear Algebra Practice ~~Midterm 2~~ Final

Spring 2019

1. (10 points) Let

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & -1 & 2 \end{pmatrix}$$

Compute AA^T and determine if it is diagonalizable. Justify your answer either with a clear demonstration or with reference to a relevant theorem.

$$AA^T = \begin{pmatrix} 1 & 3 & 0 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 3 & -1 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 10 & -3 \\ -3 & 5 \end{pmatrix}$$

Since AA^T is symmetric, we already know it is diagonalizable with real eigenvalues, but to see it directly, we compute:

$$c(\lambda) = \det(AA^T - \lambda I)$$

$$= \det\begin{pmatrix} 10-\lambda & -3 \\ -3 & 5-\lambda \end{pmatrix}$$

$$= (10-\lambda)(5-\lambda) - 9$$

$$= \lambda^2 - 15\lambda + 41$$

$$= 0$$

distinct eigenvalues $\Rightarrow$ diagonalizable

$$\Rightarrow \lambda = \frac{15 \pm \sqrt{15^2 - 4 \cdot 1 \cdot 41}}{2} = \frac{15}{2} \pm \frac{\sqrt{225-164}}{2}$$

$$= \frac{15}{2} \pm \frac{\sqrt{61}}{2}$$

2. Using the same matrix A as in problem (1),

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & -1 & 2 \end{pmatrix}$$

answer the following questions:

(a) (5 points) What is $\det(A^T A)$?

$$\det(A^TA) = \det\left(\begin{pmatrix} 1 & 0 \\ 3 & -1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 3 & 0 \\ 0 & -1 & 2 \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} 1 & 3 & 0 \\ 3 & 10 & -2 \\ 0 & -2 & 4 \end{pmatrix}$$

add $-3\,\mathrm{I}$ to II

$$= \det\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{pmatrix}$$

add $2\,\mathrm{II}$ to III

$$= \det\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= 1 \cdot 1 \cdot 0 = \boxed{0}$$

(b) (5 points) Use the above result to determine if the columns of $A^T A$ are linearly independent.

No, not linearly independent, because $A \in GL(n, \mathbb{R}) \iff A$'s columns are lin. indep $\iff \det A \neq 0$. (this is a <u>theorem</u>.)

3. Consider the matrices $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

(a) (5 points) Find the eigenvalues of each matrix.

$\boxed{\lambda = 2}$ for both: A & B are upper triangular, so the λ's sit along their diagonal

(b) (5 points) Determine the geometric multiplicities of each eigenvalue, for each matrix. Which of A and B is diagonalizable?

A has geometric mult. of λ $= 1 < 2 =$ alg. mult.

$A - 2I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has 1 pivot col., therefore $\dim E_\lambda = \dim N(A - 2I) = 1$

B has geom. mult. $2 =$ alg. mult.

$B - 2I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$; the zero matrix, whose null space is all of $\mathbb{R}^2$; therefore $\dim E_\lambda = 2$

Therefore, A is not, B is diagonalizable.

(c) (5 points) Determine whether the two matrices are similar, i.e. whether $B = PAP^{-1}$ for some $P \in \mathrm{GL}(2, \mathbb{R})$.

No, because:

B is diagonalizable (it's already diagonal!), whereas A is not.

That is, $B \sim D$ (similar to) diag. (in fact $B = D$)

but $A \not\sim D$,

whereas if $A \sim B$, then

$A \sim B \sim D \implies \underbrace{A \sim D}$

& we know this isn't poss.

4. Consider the vector $\mathbf{v} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ in $\mathbb{R}^2$.

(a) (5 points) Determine $[\mathbf{v}]_\beta$ if $\beta = \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right)$.

$$M_{\beta\sigma} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \quad M_{\sigma\beta} = M_{\beta\sigma}^{-1} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}^{-1}$$

$$= -\frac{1}{2}\begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -4 & 3 \\ 2 & -1 \end{pmatrix}$$

$$\Rightarrow [\vec{v}]_\beta = M_{\sigma\beta}\vec{v} = \frac{1}{2}\begin{pmatrix} -4 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -20+6 \\ 10-2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -14 \\ 8 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} -7 \\ 4 \end{pmatrix}}$$

(b) (5 points) Find the standard representation of the projection $\pi_{\mathbf{w}}$ onto the vector $\mathbf{w} = R_{\pi/2}\mathbf{v}$ orthogonal to $\mathbf{v}$.

$$R_{\pi/2}\vec{v} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix} = \vec{\omega}$$

$$\Rightarrow \pi_{\vec{\omega}}(\vec{e}_1) = \frac{\vec{\omega}\cdot\vec{e}_1}{\|\vec{\omega}\|^2}\vec{\omega} = \frac{\begin{pmatrix} -2 \\ 5 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix}}{(-2)^2+5^2}\begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

$$= \frac{-2}{29}\begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

$$\pi_{\vec{\omega}}(\vec{e}_2) = \frac{\vec{\omega}\cdot\vec{e}_2}{\|\vec{\omega}\|^2}\vec{\omega} = \frac{\begin{pmatrix} -2 \\ 5 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix}}{(-2)^2+5^2}\begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

$$= \frac{5}{29}\begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

$$\Rightarrow [\pi_{\vec{\omega}}]_\sigma = \frac{1}{29}\left(-2\begin{pmatrix} -2 \\ 5 \end{pmatrix} \quad 5\begin{pmatrix} -2 \\ 5 \end{pmatrix} \right) = \boxed{\frac{1}{29}\begin{pmatrix} 4 & -10 \\ -10 & 25 \end{pmatrix}}$$

(c) (5 points) Find the standard representation of the reflection R_ℓ across $\ell = \text{span}(\mathbf{v})$.

Since $R_\ell(\vec{v}) = \vec{v}$, where $\vec{v} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

and $R_\ell(\vec{\omega}) = -\vec{\omega}$, $\vec{\omega} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$,

letting $\beta = (\vec{v}, \vec{\omega}) = \left(\begin{pmatrix} 5 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 5 \end{pmatrix}\right)$,

$$[R_\ell]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and therefore

$$[R_\ell]_\sigma = M_{\beta\sigma} [R_\ell]_\beta M_{\beta\sigma}^{-1}$$

$$= \underbrace{\begin{pmatrix} 5 & -2 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}\begin{pmatrix} 5 & -2 \\ 2 & 5 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 5 & 2 \\ 2 & -5 \end{pmatrix}\left(\frac{1}{29}\begin{pmatrix} 5 & 2 \\ -2 & 5 \end{pmatrix}\right)$$

$$\boxed{= \frac{1}{29}\begin{pmatrix} 21 & 20 \\ 20 & -21 \end{pmatrix}}$$

5. Let

$$A = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$$

(a) (5 points) Find the eigenvalues of A.

$$\det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{pmatrix}$$

$$= (2-\lambda)(-6-\lambda) - 9$$

$$= \lambda^2 + 4\lambda - 21$$

$$= (\lambda + 7)(\lambda - 3)$$

$$= 0$$

$$\Longrightarrow \boxed{\lambda = -7, 3}$$

(b) (5 points) Find corresponding eigenvectors of A.

① $\underline{\lambda = -7}$: $A + 7I = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} \xrightarrow{\frac{1}{3}I} \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$

$$\xrightarrow{-I+II} \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}$$

Let $y = t$, then $x = -\frac{1}{3}t \Longrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix}$

so choosing $t = 3$, $\boxed{\vec{v}_1 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}}$

② $\underline{\lambda = 3}$: $A - 3I = \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix} \xrightarrow{(-1)I} \begin{pmatrix} 1 & -3 \\ 3 & -9 \end{pmatrix} \xrightarrow{-3I+II} \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}$

Let $y = t$, then $x = 3t \Longrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} 3 \\ 1 \end{pmatrix}$,

so $\boxed{\vec{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}}$

(c) (5 points) Is A *orthogonally* diagonalizable? Justify your answer with a clear demonstration or with reference to a relevant theorem.

Correction: Yes because A is symmetric iff it is orthogonally diagonalizable! This is a theorem (Thm. 7, p. 19 of `Symmetric…' notes). I didn't heed my own warning at the bottom of p. 17 of those notes, and therefore I lose one point!

Yes, because $A \in \mathrm{Sym}_2(\mathbb{R})$, & therefore normal, $A^TA = AA^T$. Normal $\Leftrightarrow$ orthogonally diagonalizable is a theorem.

(d) (5 points) Let

$$D = [A]_\beta = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

be the diagonal matrix with the eigenvalues of A, λ_1 and λ_2, you got in part (b), whose corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ you got in part (c). Use the fact that $A = PDP^{-1}$, and therefore $e^A = e^{PDP^{-1}} = Pe^DP^{-1}$, to compute e^A.

Here, of course, $e^A \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{1}{n!}A^n$.

$$e^A = e^{PDP^{-1}}$$

$$= Pe^DP^{-1}$$

$$= Pe^{\begin{pmatrix} -7 & 0 \\ 0 & 3 \end{pmatrix}}P^{-1}$$

$$= P\begin{pmatrix} e^{-7} & 0 \\ 0 & e^3 \end{pmatrix}P^{-1}$$

$$P = (\vec{v}_1\ \vec{v}_2) = \begin{pmatrix} -1 & 3 \\ 3 & 1 \end{pmatrix}$$

$$P^{-1} = -\frac{1}{10}\begin{pmatrix} 1 & -3 \\ -3 & -1 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} -1 & 3 \\ 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} e^{-7} & 0 \\ 0 & e^3 \end{pmatrix}\frac{1}{10}\begin{pmatrix} -1 & 3 \\ 3 & 1 \end{pmatrix}$$

$$= \frac{1}{10}\begin{pmatrix} -e^{-7} & 3e^3 \\ 3e^{-7} & e^3 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 3 & 1 \end{pmatrix}$$

$$\boxed{= \frac{1}{10}\begin{pmatrix} e^{-7}+9e^3 & -3e^{-7}+3e^3 \\ -3e^{-7}+3e^3 & 9e^{-7}+e^3 \end{pmatrix}}$$

6. (5 points) Show that $A = \begin{pmatrix} 1 & 2-i \\ 2+i & 3 \end{pmatrix}$ is Hermitian.

$$A^* = \bar{A}^T = \begin{pmatrix} \bar{1} & \overline{2+i} \\ \overline{2-i} & \bar{3} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2-i \\ 2+i & 3 \end{pmatrix}$$

$$= A$$

7. (5 points) Let $A \in \mathrm{O}(n)$ be an $n \times n$ orthogonal matrix. Show that

$$(A\mathbf{x}) \cdot (A\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. *Hint: the dot product $\mathbf{x} \cdot \mathbf{y}$ can be computed in terms of matrices,*

$$\mathbf{x} \cdot \mathbf{y} = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \mathbf{x}^T \mathbf{y}$$

$$A \in O(n) \overset{\text{def}}{\Longleftrightarrow} A^T A = I$$

$$\Longleftrightarrow (A\vec{x}) \cdot (A\vec{y})$$

$$= (A\vec{x})^T (A\vec{y})$$

$$= (\vec{x}^T A^T)(A\vec{y})$$

$$= \vec{x}^T (A^T A) \vec{y}$$

$$= \vec{x}^T I \vec{y}$$

$$= \vec{x}^T \vec{y}$$

$$= \vec{x} \cdot \vec{y}$$

8. (5 points) Show that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have

$$\mathbf{x} \cdot \mathbf{y} = \frac{1}{4}\|\mathbf{x} + \mathbf{y}\|^2 - \frac{1}{4}\|\mathbf{x} - \mathbf{y}\|^2$$

This is the **polar identity**.[1]

Start with the RHS:

$$\frac{1}{4}\|\vec{x}+\vec{y}\|^2 - \frac{1}{4}\|\vec{x}-\vec{y}\|^2$$

$$= \frac{1}{4}\left((\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y})\right) - \frac{1}{4}\left((\vec{x}-\vec{y})\cdot(\vec{x}-\vec{y})\right)$$

$$= \frac{1}{4}\left(\cancel{\vec{x}\cdot\vec{x}} + 2\vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{y}\right)$$

$$- \frac{1}{4}\left(\cancel{\vec{x}\cdot\vec{x}} - 2\vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{y}\right)$$

$$= \cancel{\frac{1}{4}\left(\vec{x}\cdot\vec{x} - \vec{x}\cdot\vec{x}\right)} + \frac{1}{4}\left(2\vec{x}\cdot\vec{y} - (-2\vec{x}\cdot\vec{y})\right)$$

$$+ \frac{1}{4}\cancel{\left(\vec{y}\cdot\vec{y} - \vec{y}\cdot\vec{y}\right)}$$

$$= \frac{1}{4} 4\, \vec{x}\cdot\vec{y}$$

$$= \vec{x}\cdot\vec{y}$$

[1] There is also a complex polar identity in $\mathbb{C}^n$, in terms of the Hermitian inner product,

$$\mathbf{z} \cdot \mathbf{w} = \frac{1}{4}\sum_{k=1}^{4} \|\mathbf{z} + i^k \mathbf{w}\|^2$$

which it would be a worthwhile exercise to try to prove.

9. (10 points) Determine if $\mathbf{w}$ lies in the span of $\mathbf{u}$ and $\mathbf{v}$. If so, write $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

$$\mathbf{u} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 6 \\ 16 \\ -5 \end{pmatrix}$$

$$\det\begin{pmatrix} | & | & | \\ \vec{u} & \vec{v} & \vec{w} \\ | & | & | \end{pmatrix} = \det\begin{pmatrix} 0 & 2 & 6 \\ 2 & 2 & 16 \\ -1 & 0 & -5 \end{pmatrix}$$

$$= 0\cdot\det\begin{pmatrix} 2 & 16 \\ 0 & -5 \end{pmatrix} - 2\det\begin{pmatrix} 2 & 16 \\ -1 & -5 \end{pmatrix} + 6\det\begin{pmatrix} 2 & 2 \\ -1 & 0 \end{pmatrix}$$

$$= -2(-10+16) + 6(0+2)$$

$$= -12 + 12$$

$$= 0$$

So, no, not linearly indep. Let's find $a, b \in \mathbb{R}$ such that $a\vec{u} + b\vec{v} = \vec{w}$:

$$a\vec{u} + b\vec{v} = \vec{w} \iff \begin{pmatrix} | & | \\ \vec{u} & \vec{v} \\ | & | \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{matrix} | \\ \vec{w} \\ | \end{matrix}$$

$$\iff \begin{pmatrix} 0 & 2 \\ 2 & 2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 6 \\ 16 \\ -5 \end{pmatrix}$$

$$\iff \left(\begin{array}{cc|c} 0 & 2 & 6 \\ 2 & 2 & 16 \\ -1 & 0 & -5 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 2 & 2 & 16 \\ 0 & 2 & 6 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 2 & 6 \\ 0 & 2 & 6 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right) \rightarrow \boxed{a=5,\ b=3}$$

Verify: $\boxed{5\vec{u} + 3\vec{v} = 5\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + 3\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 16 \\ -5 \end{pmatrix} = \vec{w} \checkmark}$

10. (10 points) Use the following properties of the determinant,

- $\det(AB) = (\det A)(\det B)$
- $\det(A^T) = \det A$

to prove that for any square matrix $A \in M_n(\mathbb{R})$ we have

$$|\det A| = \sqrt{\det(A^T A)}$$

The RHS of this equation is sometimes taken for the definition of the volume of the parallelpiped spanned by the columns of A. What's even nicer, this also works for non-square $m \times n$ matrices, with $m > n$. For example, a 3×2 matrix with two columns of size 3, would give the *area* of the parallelogram spanned by its columns *in* $\mathbb{R}^3$.

$$\det(A^T A) = (\det(A^T))(\det A)$$
$$= (\det A)^2$$

$$\Rightarrow \quad |\det A| = \sqrt{\det(A^T A)}$$

11. (10 points) The **trace** of a square matrix $A \in M_n(\mathbb{R})$ is the sum of its diagonal entries,

$$\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}$$

and we know that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. It follows from this that if A is diagonalizable, then $\operatorname{tr} A = \operatorname{tr} D = \sum_{i=1}^{n} \lambda_i$. Use this and the fact that any real or complex matrix is diagonalizable over $\mathbb{C}$ to show that

$$\det e^A = e^{\operatorname{tr} A}$$

[*Hint: use the diagonalizability of A in combination with the exponential,* $e^A = e^{PDP^{-1}} = Pe^D P^{-1}$.]

First notice that

$$e^D = \sum_{n=0}^{\infty} \frac{1}{n!} D^n = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} \lambda_1^n & & 0 \\ & \ddots & \\ 0 & & \lambda_n^n \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{n=0}^{\infty} \frac{1}{n!} \lambda_1^n & & 0 \\ & \ddots & \\ 0 & & \sum_{n=0}^{\infty} \frac{1}{n!} \lambda_n^n \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix}$$

$$\Rightarrow \det e^A = \det\left(e^{PDP^{-1}}\right) = \det\left(Pe^D P^{-1}\right)$$

$$= (\cancel{\det P})(\det e^D)(\cancel{\det P})^{-1}$$

$$= \det e^D$$

$$= \det \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix}$$

$$= e^{\lambda_1} e^{\lambda_2} \cdots e^{\lambda_n}$$

$$= e^{\sum_{i=1}^{n} \lambda_i}$$

$$= e^{\operatorname{tr} D} = e^{\operatorname{tr} A}$$

12. (5 points) **True or False**: If $A \in \mathrm{Skew}_n(\mathbb{R})$, then $\operatorname{tr} A = 0$.

Yes, true: $A^T = -A \Rightarrow a_{ii} = -a_{ii}$ for all i

$\Rightarrow a_{ii} = 0$

$\Rightarrow \operatorname{tr} A = \sum_{i=1}^{n} a_{ii} = 0$

13. (5 points) **True or False**: If A is diagonalizable with characteristic polynomial factoring as

$$c_A(\lambda) = \det(A - \lambda I) = (-1)^n(\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_k)^{n_k}$$

where $\lambda_1, \ldots, \lambda_k$ are A's distinct eigenvalues, then

$$\det A = \lambda_1^{n_1}\lambda_2^{n_2} \cdots \lambda_k^{n_k}$$

True: $A = PDP^{-1} = P\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{pmatrix}P^{-1}$ (λ_1 occurs n_1 times, λ_k occurs n_k times)

$\Rightarrow \det A = \det D$

$= \lambda_1^{n_1} \cdots \lambda_k^{n_k}$

14. (5 points) **True or False**: The standard representation of a reflection is always an orthogonal matrix.

True: Let $\vec{u}$ be a unit vector in ℓ, so $\ell = \operatorname{span}(\vec{u})$, & let $\vec{v} = R_{\pi/2}\vec{u}$, also unit length. Thus, $\beta = (\vec{u}, \vec{v})$ is an orthonormal basis, & $P = M_{\beta\sigma} \in O(2)$, & $P[R_\ell]_\beta P^{-1} = [R_\ell]_\sigma$ & $[R_\ell]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ~ diag. orthog.

15. (5 points) **True or False**: If A is diagonalizable with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, then we may write A as a sum of projections onto the corresponding eigenspaces.

$$A = \lambda_1 \pi_{E_{\lambda_1}} + \lambda_2 \pi_{E_{\lambda_2}} + \cdots + \lambda_k \pi_{E_{\lambda_k}}$$

Here, of course, and only to round things off, we mention that

$$\begin{aligned}
\pi_{E_{\lambda_i}}(\mathbf{x}) &= \pi_{\mathbf{v}_1}(\mathbf{x}) + \cdots + \pi_{\mathbf{v}_k}(\mathbf{x}) \\
&= \frac{\mathbf{v}_1 \cdot \mathbf{x}}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \cdots + \frac{\mathbf{v}_{n_i} \cdot \mathbf{x}}{\|\mathbf{v}_{n_i}\|^2}\mathbf{v}_{n_i}
\end{aligned}$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_{n_i}$ are the basis eigenvectors for E_{λ_i}.

True: $A = PDP^{-1}$, $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

not counting repetitions

$$= \begin{pmatrix} \overbrace{\lambda_1 \cdots \lambda_1}^{n_1} & & 0 \\ & \ddots & \\ 0 & & \overbrace{\lambda_k \cdots \lambda_k}^{n_k} \end{pmatrix}$$

$\Rightarrow$ $D = P^{-1}AP$ acts on

$\mathbb{R}^n = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_n}$

by multiplication by λ_i on each E_{λ_i}

$\Leftrightarrow$ $A\vec{v}_i = \lambda_i v_i = \lambda_i \dfrac{\vec{v}_i \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i} v_i$. $\Leftrightarrow$ writing: $\vec{x} = \underset{\in E_{\lambda_1}}{\vec{x}_1} + \cdots + \underset{\in E_{\lambda_k}}{\vec{x}_k}$

$\pi_{E_{\lambda_i}}(\vec{x})$ $A\vec{x} = \sum_i Ax_i = \sum_i \lambda_i x_i$

16. (5 points) Show that if U and V are two subpsaces of $\mathbb{C}^n$, then so is their sum

$$U + V \stackrel{\text{def}}{=} \{\mathbf{u} + \mathbf{v} \mid \mathbf{u} \in U \text{ and } \mathbf{v} \in V\}$$

(1) $\vec{0} \in U$, $\vec{0} \in V$ bec. U & V are subspaces

$\Longrightarrow \vec{0} = \vec{0} + \vec{0} \in U + V$

(2) $\vec{x} = \vec{u}_1 + \vec{v}_1$, $\vec{y} = \vec{u}_2 + \vec{v}_2 \in U + V$

$$\Longrightarrow \vec{x} + \vec{y} = (\vec{u}_1 + \vec{v}_1) + (\vec{u}_2 + \vec{v}_2)$$

$$= \underbrace{(\vec{u}_1 + \vec{u}_2)}_{\in U} + \underbrace{(\vec{v}_1 + \vec{v}_2)}_{\in V} \quad \text{(associativity)}$$

bec. U & V are subspaces

$\in U + V$ by the def. of $U + V$.

(3) $\vec{x} = \vec{u} + \vec{v} \in U + V$, $c \in \mathbb{C} \Longrightarrow$

$$c\vec{x} = \underbrace{c\vec{u}}_{\in U} + \underbrace{c\vec{v}}_{\in V} \in U + V$$